Spectral Triples for Hyperbolic Dynamical Systems.

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Introduction

A Smale space consists of a compact metric space and a homeomorphism (X, φ) with canonical expanding and contracting directions.

Smale spaces were introduced by David Ruelle and include:

- Shifts of finite type,
- Hyperbolic toral automorphisms,
- Solenoids (Bob Williams),
- Dynamical systems associated with certain substitution tiling spaces (Anderson and Putnam),
- the basic sets of Smale's Axiom A systems (Ruelle)

Ruelle further defined C^*-algebras associated with a Smale space, known as the stable and unstable algebra of a Smale space. Ruelle also described KMS states of these algebras.

Ian Putnam described the K-theory of these algebras and defined associated crossed product algebras now known as the stable and unstable Ruelle algebras.

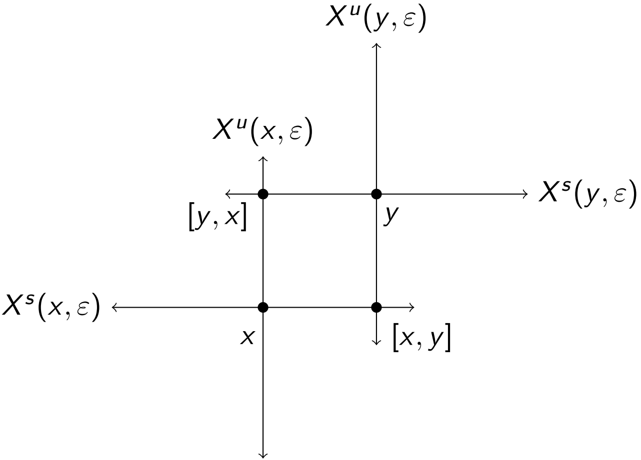
Ian Putnam and Jack Spielberg gave elegant descriptions of the above C^*-algebras, up to Morita equivalence, and we follow their development.

My goal today is to define spectral triples on these C^*-algebras.

Smale spaces

Suppose X is a compact metric space with a homeomorphism $\varphi : X \to X$. A dynamical system (X, φ) is a Smale space if there are constants $\varepsilon_X > 0$, $\lambda > 1$ such that for $x \in X$ and $0 < \varepsilon < \varepsilon_X$:

1. There are two open sets $X^s(x, \varepsilon)$ and $X^u(x, \varepsilon)$ such that
$$X^s(x, \varepsilon) \cap X^u(x, \varepsilon) = \{x\},$$
2. The product $X^s(x, \varepsilon) \times X^u(x, \varepsilon)$ is homeomorphic to a neighbourhood of x.
3. $X^s(x, \varepsilon)$ is called a local stable set and satisfies
$$\varphi(X^s(x, \varepsilon)) \subset X^s(\varphi(x), \lambda^{-1}\varepsilon),$$
4. $X^u(x, \varepsilon)$ is called a local unstable set and satisfies
$$\varphi^{-1}(X^u(x, \varepsilon)) \subset X^u(\varphi^{-1}(x), \lambda^{-1}\varepsilon).$$



The bracket map

Equivalence Relations

Given a point x in X, the stable equivalence class of x is

$$X^s(x) = \{y \in X \mid \lim_{n \to +\infty} d(\varphi^n(x), \varphi^n(y)) = 0\}.$$

The unstable equivalence class of x is

$$X^u(x) = \{y \in X \mid \lim_{n \to +\infty} d(\varphi^{-n}(x), \varphi^{-n}(y)) = 0\}.$$

Both the stable and unstable equivalence classes of a point are locally compact and Hausdorff in the topology generated by local stable and unstable sets, respectively.

Example: Hyperbolic Toral Automorphism

Let A be the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

Consider $A : \mathbb{T}^2 \to \mathbb{T}^2$ as a map on the quotient space $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. We claim that $(\mathbb{T}^2, A)$ is a Smale space.

Let $\gamma = \frac{1+\sqrt{5}}{2}$ be the golden mean and the eigenvalues for A are $\gamma > 1$ and $-\gamma^{-1}$. Now the eigenvectors for A are

$$v_s = \begin{pmatrix} 1 \\ -\gamma \end{pmatrix} \qquad \text{and} \qquad v_u = \begin{pmatrix} \gamma \\ 1 \end{pmatrix}$$

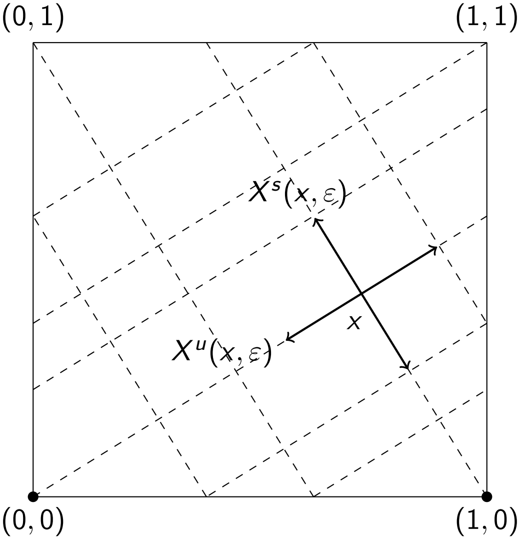
where $Av_s = -\gamma^{-1} v_s$ and $Av_u = \gamma v_u$.

Fix a point $x \in \mathbb{T}^2$, the local stable and unstable sets are

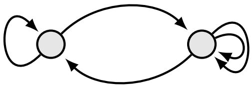
$$\begin{aligned} X^s(x,\varepsilon) &= \{x + tv_s (\operatorname{mod} \mathbb{Z}^2) \mid |t| < \varepsilon\}, \\ X^u(x,\varepsilon) &= \{x + tv_u (\operatorname{mod} \mathbb{Z}^2) \mid |t| < \varepsilon\}. \end{aligned}$$

Moreover, the global stable and unstable equivalence classes of a point x in $\mathbb{T}^2$ are defined by

$$\begin{aligned} X^s(x) &= \{x + tv_s (\operatorname{mod} \mathbb{Z}^2) \mid t \in \mathbb{R}\}, \\ X^u(x) &= \{x + tv_u (\operatorname{mod} \mathbb{Z}^2) \mid t \in \mathbb{R}\}. \end{aligned}$$


(0, 1)
(1, 1)
$X^s(x, \varepsilon)$
x
$X^u(x, \varepsilon)$
(0, 0)
(1, 0)

Example: shifts of finite type



Suppose G is a directed graph and define

$$X_G = \{\cdots e_{-2}e_{-1}.e_0e_1e_2\cdots \mid \forall i \quad e_i \text{ to } e_{i+1} \text{ is allowable}\};$$

that is, X_G consists of all possible bi-infinite paths in the graph G.

Metric on X_G:

$$d(e,f) = \inf\{2^{-n} \mid e_i = f_i \text{ for all } |i| < n\}.$$

Homeomorphism on X_G:

$$\sigma(\cdots e_{-2}e_{-1}.e_0e_1e_2\cdots) = \cdots e_{-2}e_{-1}e_0.e_1e_2\cdots$$

The dynamical system (X_G, σ) is a Smale space.

Fix a point $e \in X_G$ and let $0 < \varepsilon < \frac{1}{2}$, the local stable and unstable sets are

$$X^s(e, \varepsilon_{X_G}) = \{f \in X_G | e_i = f_i \text{ for all } i \geq 0\},$$
$$X^u(e, \varepsilon_{X_G}) = \{f \in X_G | e_i = f_i \text{ for all } i \leq 0\}.$$

Moreover, the stable and unstable equivalence classes of $e \in X_G$ are

$$X^s(e) = \{f \in X_G | e_i = f_i \text{ for some } i \geq 0\},$$
$$X^u(e) = \{f \in X_G | e_i = f_i \text{ for some } i \leq 0\}.$$

C^*-algebras of Smale spaces

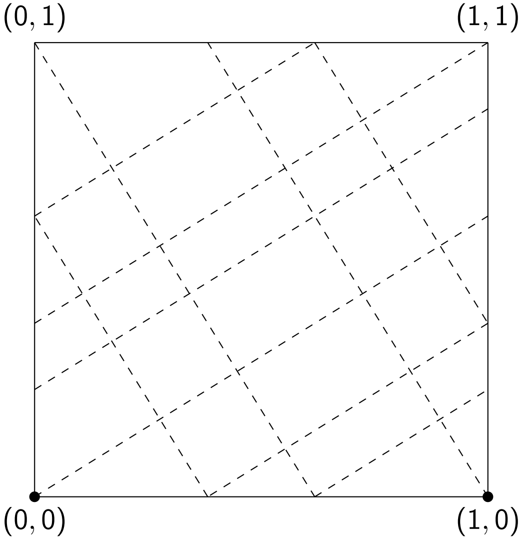
Let (X, φ) be a Smale space and let P be a finite, φ-invariant set of periodic points. Define

$$X^s(P) = \bigcup_{p \in P} X^s(p)$$

$$X^u(P) = \bigcup_{p \in P} X^u(p)$$

$$X^h(P) = X^s(P) \cap X^u(P).$$

$X^h(P)$ is countable and dense if (X, φ) is irreducible.



$X^h(\{0\})$

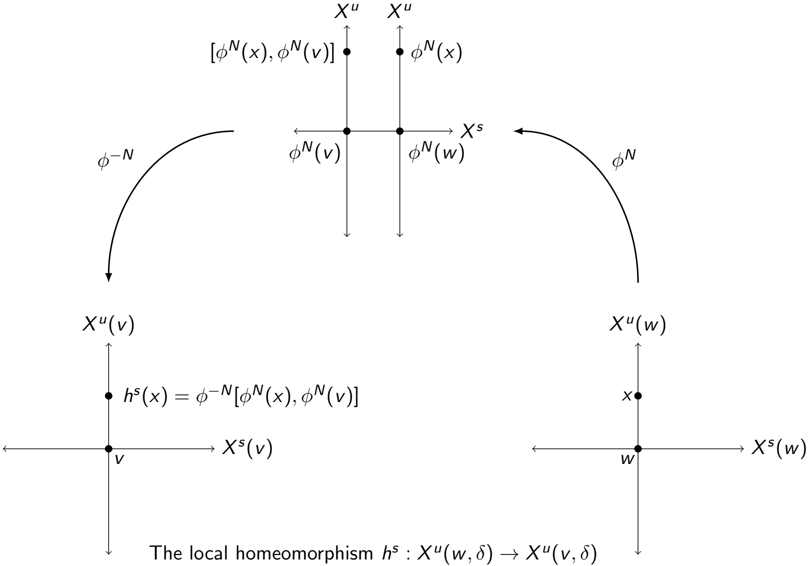
Stable and unstable equivalence leads to groupoids on (X, d, φ):

$$G^s(X, \varphi, P) = \{(v, w) | v \sim_s w \text{ and } v, w \in X^u(P)\}$$
$$G^u(X, \varphi, P) = \{(v, w) | v \sim_u w \text{ and } v, w \in X^s(P)\}.$$

These groupoids are independent of P, up to groupoid equivalence, in the sense of Muhly, Renault, and Williams.

Suppose $v \sim_s w$ and $v, w \in X^u(P)$. Then there is $\delta > 0$ such that there is a local homeomorphism $h^s : X^u(w, \delta) \to X^u(v, \delta)$ (this is illustrated on the next page). This data defines a topology on $G^s(X, \varphi, P)$ and we denote basic sets in this topology by $V^s(v, w, h^s, \delta)$.

With this topology, these groupoids are étale.



The local homeomorphism $h^s : X^u(w, \delta) \to X^u(v, \delta)$

$C_c(G^s(X,\varphi,P))$ is a complex linear space with

- Product
$$ab(x,y) = \sum_{(x,z)\in G^s(X,\varphi,P)} a(x,z)b(z,y)$$
- Involution
$$a^*(x,y) = \overline{a(y,x)}.$$

Define a Hilbert space $\mathcal{H} = \ell^2(X^h(P))$ and represent $C_c(G^s(X,\varphi,P))$ as bounded operators on $\mathcal{H}$ via

$$(a\xi)(x) = \sum_{(x,y)\in G^s(X,\varphi,P)} a(x,y)\xi(y).$$

The stable C^*-algebra $S(X,\varphi,P)$ is defined to be the completion of $C_c(G^s(X,\varphi,P))$ in the operator norm of $\mathcal{H}$. Moreover, $S(X,\varphi,P)$ is independent of P up to Morita equivalence.

Suppose a is a function in $C_c(G^s(X, \varphi, P))$ with support on the basic set $V^s(v, w, h^s, \delta)$.

Let $\{\delta_x \mid x \in X^h(P)\}$ denote the usual basis of Dirac delta functions in $\mathcal{H} = \ell^2(X^h(P))$.

Then,

$$\pi(a)\delta_x = a(h^s(x), x)\delta_{h^s(x)}$$

if $x \in X^u(w, \delta)$ and zero otherwise.

Define $Source(a) \subseteq X^u(w, \delta)$ to be the points for which a is non-zero on the source of the domain of a.

Spectral Triples

Definition

A *spectral triple* $(A, \mathcal{H}, D)$ consists of

(i) a separable Hilbert space $\mathcal{H}$,

(ii) a $*$-algebra A of bounded operators on $\mathcal{H}$,

(iii) an unbounded self-adjoint operator D on $\mathcal{H}$ such that:

- (a) the set $\{a \in A | [D, a] \in \mathcal{B}(\mathcal{H})\}$ is norm dense in A and
- (b) the operator $a(1 + D^2)^{-1}$ is a compact operator on $\mathcal{H}$ for all a in A.

Definition

Suppose $(A, \mathcal{H}, D)$ is a spectral triple over a unital C^*-algebra A with

$$\mathrm{Tr}((1+D^2)^{-\frac{p}{2}}) < \infty$$

for some positive number p. Then the spectral triple is said to be *p-summable*. Furthermore, the value

$$\dim_S((A, \mathcal{H}, D)) := \inf\{p > 0 | \mathrm{Tr}((1+D^2)^{-\frac{p}{2}}) < \infty\}$$

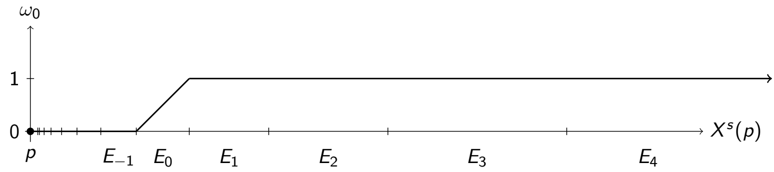
is called the *spectral dimension* of the spectral triple. We call $(A, \mathcal{H}, D)$ *θ-summable* if, for all $t > 0$,

$$\mathrm{Tr}(e^{-t(1+D^2)}) < \infty.$$

Let us consider $S(X, \varphi, P)$ as operators on $\mathcal{H}$ so that we may omit the representation.

Think of our φ-invariant set of periodic points P as a single fixed point.

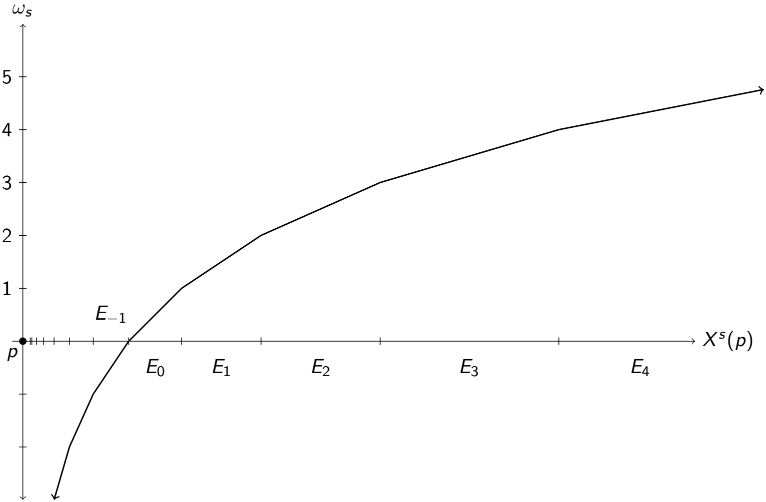
Idea: Take $x \in X^s(P)$ and count how many iterations of φ it requires to get $\varphi^n(x)$ in a fixed neighbourhood of P.



Define a function $\omega_0 : X^s(P) \to [0,1]$ as above.

The function ω_0 leads to a function $\omega : X^s(P) \setminus P \to \mathbb{R}$ via

$$\omega_s(x) = \sum_{n=0}^{\infty} \omega_0 \circ \varphi^n(x) - \sum_{n=1}^{\infty} (1 - \omega_0) \circ \varphi^{-n}(x).$$


ω_s
5
4
3
2
1
E_{-1}
p
$X^s(p)$
E_0
E_1
E_2
E_3
E_4

Lemma

Suppose P is a finite, φ-invariant set of periodic points in a Smale space (X, φ) and $\omega : X^s(P) \setminus P \to \mathbb{R}$ is defined as above. Let $a \in C_c(G^s(X, \varphi, P))$ be supported on a basic set $V^s(v, w, h^s, \delta)$ so that $Source(a) \subseteq X^u(w, \delta)$ and $h^s(Source(a)) \subseteq X^u(v, \delta)$. Then,

1. *there exists $N \in \mathbb{Z}$ such that $E_n \cap Source(a) = \emptyset$ for all $n \leq N$,*
2. *for all N, the number of points in $E_N \cap Source(a)$ is finite,*
3. *if $x \in E_N \cap Source(a)$, then there exists $K \in \mathbb{N}$ such that $h^s(x) \in \bigcup_{k=-K}^{K} E_{N+k}$.*

Define a self-adjoint, unbounded operator D on $\mathcal{H} = \ell^2(X^h(P))$ by

$$D\delta_x = \omega(x)\delta_x.$$

Theorem

$(S, \mathcal{H}, D)$ is a non-unital, θ-summable spectral triple.

Proof of bounded commutator.

Let $a \in C_c(G^s(X, \varphi, P))$ be supported on a basic set $V^s(v, w, h^s, \delta)$ so that $Source(a) \subseteq X^u(w, \delta)$ and $h^s(Source(a)) \subseteq X^u(v, \delta)$. Then,

$$\begin{aligned}
\|[D, a]\delta_x\| &= \|(\omega(h^s(x))a(h^s(x), x) - \omega(x)a(h^s(x), x))\delta_{h^s(x)}\| \\
&\leq |\omega(h^s(x)) - \omega(x)||a(h^s(x), x)| \\
&\leq |(N + K + 1) - N)||a(h^s(x), x)| = (K + 1)|a(h^s(x), x)|.
\end{aligned}$$

Much more desirable for a spectral triple to be finitely summable (so that we can work out the spectral dimension).

Let ω_0 be (locally) Lipschitz continuous. Then it follows that $\omega : X^s(P) \setminus P \to \mathbb{R}$ is locally Lipschitz continuous as well.

Recall that $\lambda > 1$ is the expansive constant of the Smale space (X, φ).

Define a self-adjoint, unbounded operator $\mathbb{D}$ on $\mathcal{H} = \ell^2(X^h(P))$ by

$$\mathbb{D}\delta_x = \lambda^{\omega(x)}\delta_x.$$

Theorem

$(S, \mathcal{H}, \mathbb{D})$ is a summable, non-unital spectral triple. The spectral dimension of $(S, \mathcal{H}, \mathbb{D})$ is $\log_\lambda(e)h(X)$, where $h(X)$ is the topological entropy of (X, φ).

Thank-you!

References:

- P. S. Muhly, J. N. Renault and D. P. Williams, *Equivalence and isomorphism for groupoid C^*-algebras*, J. Operator Theory, **17** (1987), 3-22.
- I.F. Putnam, *C^*-Algebras from Smale Spaces*, Canad. J. Math. **48** (1996), 175-195.
- I.F. Putnam and J. Spielberg *The Structure of C^*-Algebras Associated with Hyperbolic Dynamical Systems*, J. Func. Analysis **163** (1999), 279-299.
- D. Ruelle, *Noncommutative Algebras for Hyperbolic Diffeomorphisms*, Invent. Math. **93** (1988), 1-13.
- S. Smale, *Differentiable Dynamical Systems*, Bull. Amer. Math. Soc. **73** (1967), 747-817.
- M.F. Whittaker, *Spectral Triples for Hyperbolic Dynamical Systems*, Pre-print: arXiv 1011.3292, 2010.