K-Theoretic Duality for Smale Spaces.

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In the beginning...

Poincaré Duality: Suppose M is a compact, oriented, n-dimensional manifold with $H^k(M)$ and $H_k(M)$ denoting the kth cohomology group and the kth homology group of M respectively. For all $k \in \mathbb{Z}$, there is a natural isomorphism

$$\begin{aligned} D : H^k(M) &\to H_{n-k}(M) \\ \alpha &\mapsto \alpha \cap \Delta, \end{aligned}$$

where Δ is the fundamental class of M.

- Originally formulated in 1893 by Henri Poincaré in terms of Betti numbers:

  The kth and $(n - k)$th Betti numbers of an n-dimensional compact manifold are equal.

- Modern formulation, in terms of cohomology and the cap product due to Eduard Čech and Hassler Whitney.
- Poincaré duality is natural in the sense that if $f : M \to M'$ is continuous and orientation preserving, then

$$D' = f_* D f^*$$

  where f_* and f^* are the induced maps on homology and cohomology respectively.

Kasparov's bivariant K-theory

- Let A and B be C^*-algebras, Kasparov defines an abelian group $KK(A, B)$ which is contravariant in the first variable and covariant in the second.
- KK-theory simultaneously defines K-theory and K-homology:

$$KK^*(A, \mathbb{C}) \cong K^*(A)$$
$$KK^*(\mathbb{C}, A) \cong K_*(A).$$

- An extension of a C^*-algebra A,

$$0 \longrightarrow \mathcal{K}(\mathcal{H}) \longrightarrow E \longrightarrow A \longrightarrow 0,$$

defines a K-homology class in $KK^1(A, \mathbb{C})$

Facts about KK-theory

- A $*$-homomorphism $\alpha : A \to B$ defines a class $[\alpha]$ in $KK(A, B)$.
- There is a product in KK-theory:

$$x \otimes_D y : KK(A, D) \otimes KK(D, B) \to KK(A, B).$$

- If $\alpha : A \to D$ and $\beta : D \to B$ then the product is given by

$$[\alpha] \otimes_D [\beta] = [\beta \circ \alpha].$$

K-theoretic duality

Definition [Kasparov, Connes, Kaminker-Putnam]: Let A and B be nuclear C^*-algebras and consider the tensor product $A \otimes B$. Suppose there is a K-homology class $\Delta \in KK^1(A \otimes B, \mathbb{C})$ and a K-theory class $\delta \in KK^1(\mathbb{C}, A \otimes B)$ satisfying

$$\begin{aligned} &\text{(i)} \qquad \delta \otimes_B \Delta = \ [1_A], \\ &\text{(ii)} \qquad \delta \otimes_A \Delta = -[1_B], \end{aligned}$$

then A and B are said to be **Poincaré dual**.

- Poincaré duality induces group isomorphisms from the K-theory of A to the K-homology of B:

$$\cdot \otimes_A \Delta : K_i(A) \to K^{i+1}(B)$$
$$\delta \otimes_B \cdot : K^i(B) \to K_{i+1}(A).$$

- Suppose $[p] \in K_0(A)$, then we have
    - $[p] \otimes [1_B] \otimes [1_{C_0(\mathbb{R})}] \in KK(B \otimes C_0(\mathbb{R}), A \otimes B \otimes C_0(\mathbb{R}))$
    - $\Delta \in KK(A \otimes B \otimes C_0(\mathbb{R}), \mathbb{C})$

  so that

$$[p] \otimes_A \Delta \in KK(B \otimes C_0(\mathbb{R}), \mathbb{C}) = KK^1(B, \mathbb{C}) = K^1(B).$$

Examples

- Kasparov, Connes - Skandalis (1980s): The first examples of K-theoretic duality were given between $A = C(M)$ and $B = C_0(TM)$.
- Connes (1994) The first noncommutative example was given between A_θ and A_θ^{op}.
- Kaminker-Putnam (1996) The first odd duality was defined between the stable and unstable Ruelle algebras of a shift of finite type.
- Emerson (2001) An odd duality was given for $\partial\Gamma \rtimes \Gamma$ where Γ is a hyperbolic group satisfying certain hypotheses, and $\partial\Gamma$ is its Gromov boundary.

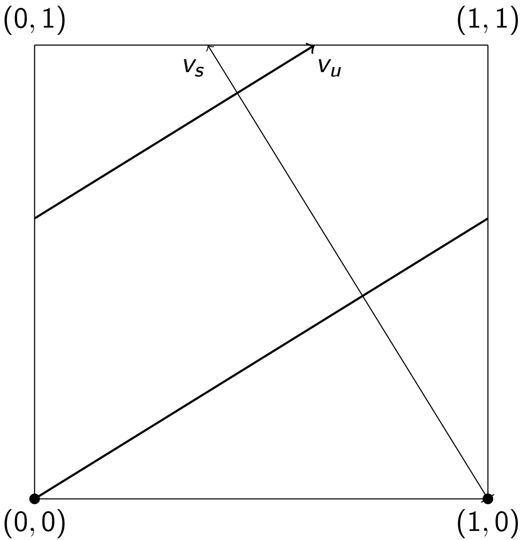
Hyperbolic Toral Automorphism

Let A be the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$

- $A : \mathbb{T}^2 \to \mathbb{T}^2$ is a homeomorphism, where $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$.
- Let $\gamma = \frac{1+\sqrt{5}}{2}$ be the golden mean.
- The eigenvalues for A are $\gamma > 1$ and $-\gamma^{-1}$ corresponding with eigenvectors

$$v_s = \begin{pmatrix} 1 \\ -\gamma \end{pmatrix} \qquad \text{and} \qquad v_u = \begin{pmatrix} \gamma \\ 1 \end{pmatrix}.$$



Hyperbolic Toral Automorphism

- Let x, y be in $\mathbb{T}^2$
    - Stable equivalence:

$$x \sim_s y \iff \lim_{n\to\infty} d(A^n(x), A^n(y)) = 0.$$

      Let $X^s(x)$ denote the stable equivalence class of x and observe:

$$X^s(x) = \{x + tv_s(\text{mod}\mathbb{Z}^2) \mid t \in \mathbb{R}\}.$$

      Moreover, for $\varepsilon > 0$, the local stable equivalence class of x:

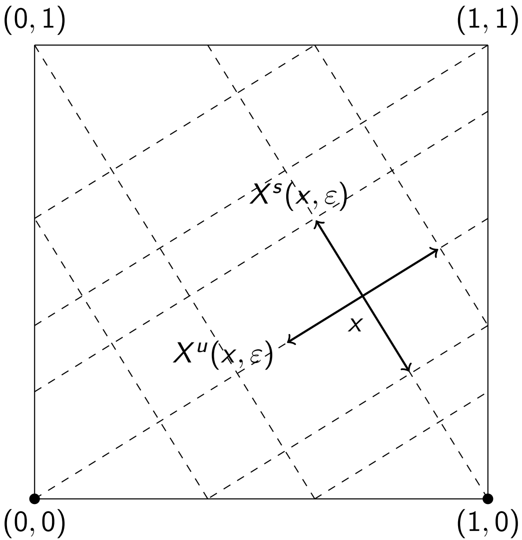
$$X^s(x, \varepsilon) = \{x + tv_s(\text{mod}\mathbb{Z}^2) \mid |t| < \varepsilon\}.$$

    - Similarly, unstable equivalence:

$$x \sim_u y \iff \lim_{n\to\infty} d(A^{-n}(x), A^{-n}(y)) = 0$$
$$X^u(x) = \{x + tv_u(\text{mod}\mathbb{Z}^2) \mid t \in \mathbb{R}\}$$
$$X^s(x, \varepsilon) = \{x + tv_u(\text{mod}\mathbb{Z}^2) \mid |t| < \varepsilon\}.$$

    - Homoclinic equivalence:

$$x \sim_h y \iff x \sim_s y \text{ and } x \sim_u y.$$



Local and global equivalence classes of x

Groupoids (Ruelle, Putnam-Spielberg)

- Notice that $\bar{0} = (0,0)$ is a fixed point of A.
- Stable groupoid:

$$G^s = \{(v,w) | v \sim_s w \text{ and } v,w \in X^u(\bar{0})\}.$$

Suppose $(v,w) \in G^s$, a nbhd base for the topology is given by a local homeomorphism

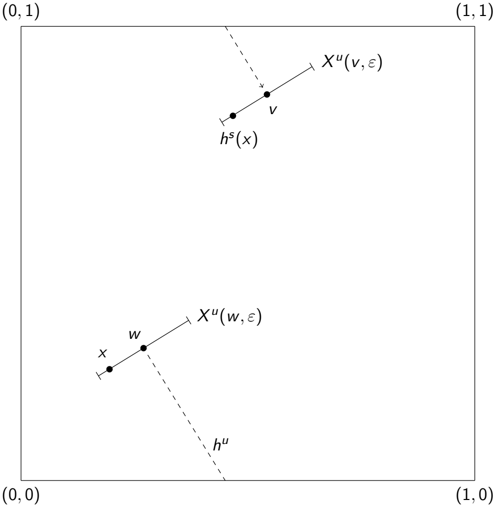
$$h^s : X^u(w,\varepsilon) \to X^u(v,\varepsilon).$$

- Unstable groupoid:

$$G^u = \{(v,w) | v \sim_u w \text{ and } v,w \in X^s(\bar{0})\}$$

Suppose $(v,w) \in G^u$, a nbhd base for the topology is given by a local homeomorphism

$$h^u : X^s(w,\varepsilon) \to X^s(v,\varepsilon).$$



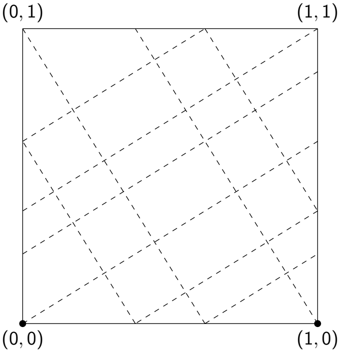
The local homeomorphism $h^s : X^u(w, \varepsilon) \to X^u(v, \varepsilon)$

C^*-algebras (Ruelle)

- Consider the Hilbert space

$$\mathcal{H} = \ell^2(X^h(\bar{0})),$$

where $X^h(\bar{0}) = \{y \mid y \sim_s \bar{0} \text{ and } y \sim_u \bar{0}\}$, which is countable and discrete.



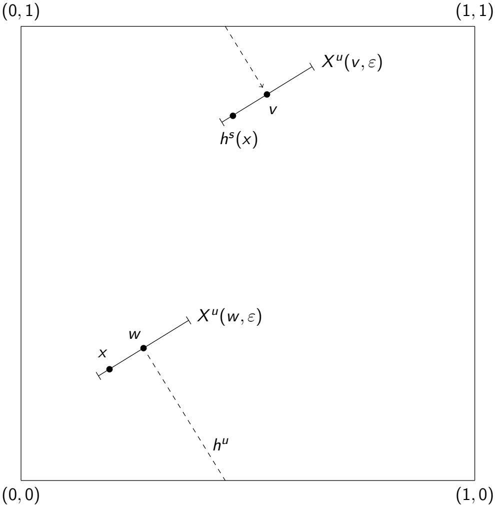
$X^h(\bar{0})$

- There are representations of the algebras $C_c(G^s)$ and $C_c(G^u)$ on the Hilbert space $\mathcal{H}$, for the stable groupoid we have:

$$(\pi_s(a)\xi)(x) = \sum_{(x,y)\in G^s} a(x,y)\xi(y).$$

- The completions, in the operator norm, of these algebras are known as the stable and unstable C^*-algebras S and U.
- These algebras have canonical generating sets consisting of functions supported on the nbhd base of G^s and G^u, respectively.
- Given a delta function $\delta_x \in \mathcal{H}$ and a function $a \in S$ with support on a basic set we have

$$\pi_s(a)\delta_x = \begin{cases} a(h^s(x), x)\delta_{h^s(x)} & \text{if } x \in X^u(w,\varepsilon) \\ 0 & \text{otherwise.} \end{cases}$$



The local homeomorphism $h^s : X^u(w, \varepsilon) \to X^u(v, \varepsilon)$

Ruelle Algebras (Putnam)

- The homeomorphism φ induces an automorphism on S by

$$\alpha_s(a)(x, y) = a(A^{-1}(x), A^{-1}(y))$$

where a is in S and (x, y) are in G^s.

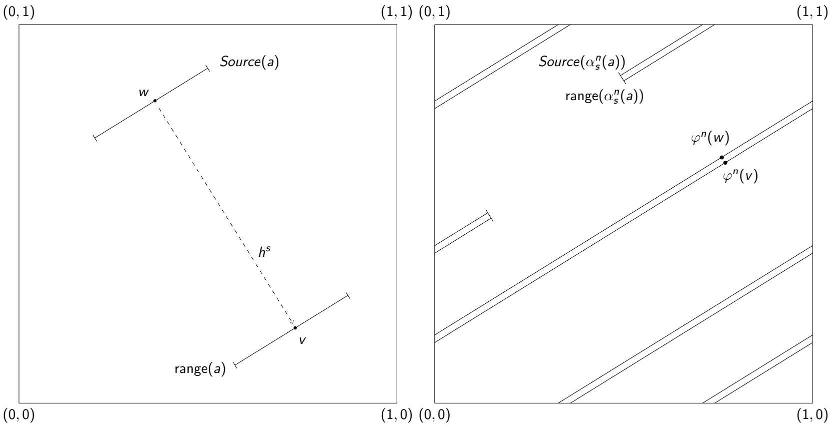
- The homeomorphism φ also induces a canonical unitary on the Hilbert space $\mathcal{H}$ via $u\delta_x = \delta_{A(x)}$.
- The pair (π_s, u) are covariant for S and we obtain the cross product

$$R^s := S \rtimes_{\alpha_s} \mathbb{Z},$$

which is called the stable Ruelle algebra.

- Similarly, the unstable Ruelle algebra is the crossed product

$$R^u := U \rtimes_{\alpha_u} \mathbb{Z}.$$



$a \in S$

$\alpha_s^n(a) \in S$ where n is some (large) positive integer

Theorem (Kaminker-Putnam-W): The stable and unstable Ruelle algebras R^s and R^u are Poincaré dual.

- To prove this we must construct a K-theory class $\delta \in KK^1(\mathbb{C}, R^s \otimes R^u)$ and a K-homology class $\Delta \in KK^1(R^s \otimes R^u, \mathbb{C})$, then show that they satisfy

$$\begin{aligned} &\text{(i)} \qquad \delta \otimes_{R^u} \Delta = \ [1_{R^s}], \\ &\text{(ii)} \qquad \delta \otimes_{R^s} \Delta = -[1_{R^u}]. \end{aligned}$$

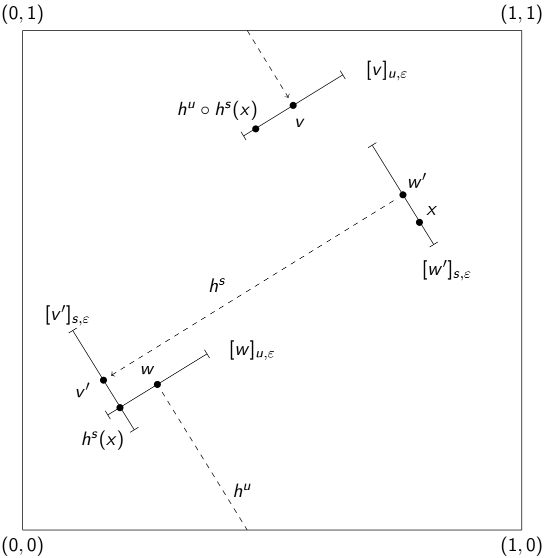
The K-homology class

To construct a class $\Delta \in KK^1(R^s \otimes R^u, \mathbb{C})$ we shall construct an exact sequence of C^*-algebras:

$$0 \to \mathcal{K}(\overline{\mathcal{H}}) \to \mathcal{E} \to R^s \otimes R^u \to 0.$$

Lemma (Putnam): Let $a \in S$ and $b \in U$, then we have:

1. $\alpha_s^n(a)b \in \mathcal{K}(\mathcal{H})$ for all $n \in \mathbb{N}$,
2. $\alpha_s^{-n}(a)b \to 0$ and $b\alpha_s^{-n}(a) \to 0$ as $n \to +\infty$,
3. $[\alpha_s^n(a), b] \to 0$ as $n \to +\infty$.



ab is a compact operator

- We have a problem: Compact operators are bad news if we want an extension of $R^s \otimes R^u$.
- We fix this by defining a new Hilbert space

$$\overline{\mathcal{H}} = \mathcal{H} \otimes \ell^2(\mathbb{Z}) = \bigoplus_{n \in \mathbb{Z}} \mathcal{H}.$$

Let B denote the bilateral shift on $\ell^2(\mathbb{Z})$, $B\delta_n = \delta_{n-1}$.

- Define $\overline{\pi}_s : R^s \to \mathcal{B}(\mathcal{H} \otimes \ell^2(\mathbb{Z}))$ via

$$\overline{\pi}_s(a) = \bigoplus_{n \in \mathbb{Z}} \alpha_s^n(a) \qquad \overline{\pi}_s(u) = 1 \otimes B \qquad [a \in S].$$

- Define $\overline{\pi}_u : R^u \to \mathcal{B}(\mathcal{H} \otimes \ell^2(\mathbb{Z}))$ via

$$\overline{\pi}_u(b) = b \otimes 1 \qquad \overline{\pi}_u(u) = u \otimes B^* \qquad [b \in U].$$

The K-homology class Δ

- Let $\mathcal{E} = C^*(\overline{\pi}_s(R^s), \overline{\pi}_u(R^u), \mathcal{K}(\overline{\mathcal{H}}))$, then

$$\mathcal{E}/\mathcal{K}(\overline{\mathcal{H}}) \cong R^s \otimes R^u.$$

- This gives an extension:

$$0 \to \mathcal{K}(\overline{\mathcal{H}}) \to \mathcal{E} \to R^s \otimes R^u \to 0,$$

the required class $\Delta \in KK^1(R^s \otimes R^u, \mathbb{C})$.

General Smale spaces

These arguments apply to all Smale spaces:

- Shifts of finite type
- Solenoids
- Certain substitution tiling spaces
- Smale's horeshoe

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