Poincaré Duality and Spectral Triples for Hyperbolic Dynamical Systems

by

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B.Sc., University of Victoria, 1999
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ABSTRACT

We study aspects of noncommutative geometry on hyperbolic dynamical systems known as Smale spaces. In particular, there are two C^*-algebras, defined on the stable and unstable groupoids arising from the hyperbolic dynamics. These give rise to two additional crossed product C^*-algebras known as the stable and unstable Ruelle algebras. We show that the Ruelle algebras exhibit noncommutative Poincaré duality. As a consequence we obtain isomorphisms between the K-theory and K-homology groups of the stable and unstable Ruelle algebras. A second result defines spectral triples on these C^*-algebras and we show that the spectral dimension recovers the topological entropy of the Smale space itself. Finally we define a natural Fredholm module on the Ruelle algebras in the special case that the Smale space is a shift of finite type. Using unitary operators arising from the Pimsner-Voiculescu sequence we compute the index pairing with our Fredholm module for specific examples.

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Chapter 1

Introduction

The main results in this dissertation come from aspects of noncommutative geometry on hyperbolic dynamical systems known as Smale spaces. The first exhibits a natural duality between the C^*-algebras associated to the weak stable and unstable equivalence classes of a hyperbolic dynamical system. The second defines a type of metric along these equivalence classes, which gives rise to spectral triples on the stable and unstable C^*-algebras. Finally, in the special case of a shift of finite type, we define natural Fredholm modules, and perform index computations.

Before introducing our results, we very briefly describe the setting of a Smale space and the C^*-algebras associated with a Smale space. These constructions can be found in [34, 35, 36, 37, 42, 43, 44] and are described in detail in Chapter 2 and Chapter 3.

Suppose X is a compact metric space and $\varphi : X \to X$ a homeomorphism. We say (X, φ) is a Smale Space if X is locally a product space of contracting and expanding directions with respect to φ. Smale spaces were introduced as a purely topological description of the basic sets of Axiom A diffeomorphisms on a compact manifold [42]. A basic set is an irreducible subset of the manifold but does not need to be a manifold itself. In fact, these sets are usually fractal and have no smooth structure whatsoever. Examples of Smale spaces include shifts of finite type, hyperbolic toral automorphisms, solenoids, and certain aperiodic substitution tiling systems (such as the Penrose tilings [1]).

Let (X, φ) be a Smale space. In the spirit of noncommutative geometry, four C^*-algebras are associated to (X, φ), which are constructed in several steps. First we take the groupoids given by the stable and unstable equivalence relations. The transverse nature of these equivalence relations allows us to restrict our attention to certain equivalence classes associated with finite sets of periodic points without losing any essential aspects of the groupoids. Using these restricted groupoids we define stable and unstable C^*-algebras $S(X, \varphi, Q)$ and $U(X, \varphi, P)$, where P and Q are finite φ-invariant sets of periodic points. We remark that, up to Morita equivalence, the choice of P and Q doesn't matter. The original homeomorphism φ can be extended to the stable and unstable groupoids and gives rise to automorphisms, α_s and α_u, on $S(X, \varphi, Q)$ and $U(X, \varphi, P)$ respectively. Using these automorphisms, crossed product C^*-algebras are produced which are known as the stable and unstable Ruelle algebras:

$$S(X, \varphi, Q) \rtimes_{\alpha_s} \mathbb{Z} \qquad \text{and} \qquad U(X, \varphi, P) \rtimes_{\alpha_u} \mathbb{Z}.$$

These C^*-algebras have many remarkable properties and are the setting for our noncommutative duality result.

1.1 The Duality Theorem

The duality theorem for Smale spaces relates the K-theory of $S \rtimes_{\alpha_s} \mathbb{Z}$ with the K-homology of $U \rtimes_{\alpha_u} \mathbb{Z}$ as well as relating the K-theory of $U \rtimes_{\alpha_u} \mathbb{Z}$ with the K-homology of $S \rtimes_{\alpha_s} \mathbb{Z}$. The duality theorem is a form of noncommutative Poincaré duality, a notion described by Kasparov [25] for groups acting on manifolds, Connes [8] for general C^*-algebras in the even case, and by Kaminker and Putnam [26] for general C^*-algebras in the odd case. We note that Poincaré duality was used by Connes in his study of the standard model of particle physics and in his definition of a noncommutative manifold [8].

Recall that K-theory is a covariant functor on the category of C^*-algebras and there is a dual contravariant theory called K-homology. These can be given a consistent definition using Kasparov's KK-theory, so that, for a C^*-algebra A we have

$$\begin{aligned} KK^*(A,\mathbb{C}) &= K^*(A) \quad (K-\text{homology}) \\ KK^*(\mathbb{C},A) &= K_*(A) \quad (K-\text{theory}). \end{aligned}$$

The duality theorem is the following.

1.1.1 Duality Theorem. *Let (X,φ) be an irreducible Smale Space. Then $S \rtimes_{\alpha_s} \mathbb{Z}$ and $U \rtimes_{\alpha_u} \mathbb{Z}$ are Poincaré dual; that is, there are duality classes δ in $KK^1(\mathbb{C}, S \rtimes_{\alpha_s} \mathbb{Z} \otimes U \rtimes_{\alpha_u} \mathbb{Z})$ and Δ in $KK^1(S \rtimes_{\alpha_s} \mathbb{Z} \otimes U \rtimes_{\alpha_u} \mathbb{Z}, \mathbb{C})$ such that*

$$\begin{aligned} \delta \otimes_{U \rtimes_{\alpha_u} \mathbb{Z}} \Delta &\cong 1_{S \rtimes_{\alpha_s} \mathbb{Z}} \quad \textit{and} \\ \delta \otimes_{S \rtimes_{\alpha_s} \mathbb{Z}} \Delta &\cong -1_{U \rtimes_{\alpha_u} \mathbb{Z}}. \end{aligned}$$

Given that $S \rtimes_{\alpha_s} \mathbb{Z}$ and $U \rtimes_{\alpha_u} \mathbb{Z}$ are Poincaré dual, a natural isomorphism is defined between the K-theory of $S \rtimes_{\alpha_s} \mathbb{Z}$ and the K-homology of $U \rtimes_{\alpha_u} \mathbb{Z}$ by taking the Kasparov product with Δ in $KK^1(S \rtimes_{\alpha_s} \mathbb{Z} \otimes U \rtimes_{\alpha_u} \mathbb{Z}, \mathbb{C})$. Similarly, a natural isomorphism is defined from the K-homology of $U \rtimes_{\alpha_u} \mathbb{Z}$ to the K-theory of $S \rtimes_{\alpha_s} \mathbb{Z}$ by taking the Kasparov product with δ in $KK^1(\mathbb{C}, S \rtimes_{\alpha_s} \mathbb{Z} \otimes U \rtimes_{\alpha_u} \mathbb{Z})$. Of course, the opposite is true as well; that we obtain isomorphisms between the K-theory of $U \rtimes_{\alpha_u} \mathbb{Z}$ and the K-homology of $S \rtimes_{\alpha_s} \mathbb{Z}$ in an analogous fashion. Moreover, in many cases we note that the K-homology is not at all well understood and the duality theorem gives insight into the meaning of these groups. As an example, for many aperiodic substitution tiling systems, the K-homology was previously not known.

At this point it is appropriate to describe some of the background and ideas leading up to this result. Kaminker and Putnam [26] proved a special case of the duality theorem when the Smale Space was a shift of finite type. Their construction used the fact that, for a shift of finite type, the Ruelle algebras are Morita equivalent to Cuntz-Krieger algebras $\mathcal{O}_A$ and $\mathcal{O}_{A^t}$, arising from a matrix A describing the shift of finite type. Kaminker and Putnam then used work of Evans [19] and Voiculescu [45] to define the fundamental class Δ in $K^1(\mathcal{O}_A \otimes \mathcal{O}_{A^t})$, represented by an extension of $\mathcal{O}_A \otimes \mathcal{O}_{A^t}$, using creation and annihilation operators acting on a subspace, determined by the matrix A, of the full Fock space of a finite dimensional Hilbert space. The duality isomorphism then followed from a very technical argument and a criterion for duality. We note that the proofs in

[26] are combinatorial in nature, which is typical for shifts of finite type.

Our new approach to the general duality theorem is quite geometric in nature and comes from the dynamics. The fundamental class Δ in $K^1(S \rtimes_{\alpha_s} \mathbb{Z} \otimes U \rtimes_{\alpha_u} \mathbb{Z})$ is also constructed by defining an extension

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{E} \longrightarrow S \rtimes_{\alpha_s} \mathbb{Z} \otimes U \rtimes_{\alpha_u} \mathbb{Z} \longrightarrow 0.$$

The class δ in $KK^1(\mathbb{C}, S \rtimes_{\alpha_s} \mathbb{Z} \otimes U \rtimes_{\alpha_u} \mathbb{Z})$ is defined using a $*$-homomorphism from the continuous functions on the circle to $S \rtimes_{\alpha_s} \mathbb{Z} \otimes U \rtimes_{\alpha_u} \mathbb{Z}$ and uses the transverse nature of the stable and unstable groupoids in a fundamental way. Finally, we prove that these two classes are Poincaré dual.

It would be remiss at this point to not mention that Kaminker and Putnam have an unpublished manuscript [27] in which they prove the general duality theorem for Smale spaces using Connes-Higson E-theory. Their proof uses the stable and unstable mapping cylinders of a Smale space, which we do not describe here. Many of their arguments apply to our proof, and we indicate their use with citations.

1.2 Spectral Triples

We now explore spectral triples on Smale spaces. Connes defined spectral triples as a method of extending the Atiyah-Singer Index Theorem to noncommutative spaces, which are defined as C^*-algebras. Spectral triples encode geometric data from a C^*-algebra in an analytic way. We define several related spectral triples on the C^*-algebras arising from a Smale space.

To begin, we define functions on the stable and unstable foliations of the Smale space. A periodic point in a Smale space can be viewed as an attractor along a stable foliation under iterations of φ, and an attractor along an unstable foliation under iterations of φ^{-1}. Using this picture, on the stable equivalence class of a periodic point, we define a function ω_s which essentially counts the number of iterations of φ required for a point to either enter a neighbourhood of a periodic point or the number of inverse iterations required for a point to be removed from a neighbourhood of a periodic point. Now several spectral triples are defined using these functions on $S(X, \varphi, Q)$ and $U(X, \varphi, P)$.

The first spectral triple, which we define on $S(X, \varphi, Q)$ and denote $(S, \mathcal{H}, D)$, is defined using the function ω_s directly. We show that the spectral triple is θ-summable, a notion introduced by Connes [9]. Moreover, the triple extends to the Ruelle algebra $S \rtimes_{\alpha_s} \mathbb{Z}$ so that we also have a spectral triple $(S \rtimes_{\alpha_s} \mathbb{Z}, \mathcal{H}, D)$. Similarly, we obtain spectral triples $(U, \mathcal{H}, D)$ and $(U \rtimes_{\alpha_u} \mathbb{Z}, \mathcal{H}, D)$.

Smale spaces exhibit exponential growth in the stable direction under iterations of φ^{-1} and exponential growth in the unstable direction under iterations of φ. Let $\lambda > 1$ denote this exponential growth rate, which controls the rate of local contraction, of the Smale space and define a new function as $f(x) = \lambda^{\omega_s(x)}$ on the stable foliation. We define a spectral triple using this function on $S(X, \varphi, Q)$ and denote the new triple $(S, \mathcal{H}, \mathfrak{D})$. Of course, a similar construction defines a spectral triple $(U, \mathcal{H}, \mathfrak{D})$ on the unstable algebra. While these spectral triples no longer extend to the Ruelle algebras they are finitely summable. In fact, the summability recovers the topological entropy $h(X, \varphi)$ of the Smale space itself. We note that the topological entropy $h(X, \varphi)$ of a Smale space defines a type of limiting value on the global growth rate.

1.3 Index Theory

We now turn our attention to shifts of finite type and Fredholm modules on the associated C^*-algebras. A Fredholm module defines a class in the K-homology of a C^*-algebra A and there is an index pairing $K_1(A) \times K^1(A) \to \mathbb{Z}$. Our aim in this section is to define a Fredholm module and compute the index pairing for a shift of finite type.

Given a shift of finite type and two φ-invariant, finite, disjoint sets of periodic points we define a projection, p on the Hilbert space $\mathcal{H} = \ell^2(X^h(P, Q))$, where $X^h(P, Q)$ denotes the set of points that is stably equivalent to P and unstably equivalent to Q. We then show that this projection commutes modulo compact operators with all elements of the C^*-algebra $S(X, \varphi, Q)$. This implies that $F = 2p - 1$ defines a Fredholm module $(S(X, \varphi, Q), \mathcal{H}, F)$.

Since we would like to compute an index pairing, we now turn to unitary operators in $S(X, \varphi, Q)$. Putnam has shown in [34] that $S(X, \varphi, Q)$ is an AF-algebra; that is, $S(X, \varphi, Q)$ is an inductive limit of matrix algebras and therefore $K_1(S(X, \varphi, Q)) = 0$. However, we are saved by the fact that the Fredholm module extends to the stable Ruelle algebra $S \rtimes_{\alpha_s} \mathbb{Z}$. Now using the Pimsner-Voiculescu sequence we are able to produce unitary operators on the unitization of $S \rtimes_{\alpha_s} \mathbb{Z}$. Finally, in two examples, we compute the index pairing with unitary operators we have produced and the Fredholm module $(S \rtimes_{\alpha_s} \mathbb{Z}, \mathcal{H}, F)$.

We believe that these are the first index computations on a shift of finite type and hope that this invariant will give insight into their structure. We also note that, due to Kaminker and Putnam's duality theorem, we have an isomorphism from the K-theory groups of the unstable Ruelle algebra to the K-homology groups of the stable Ruelle algebra. Therefore, all odd Fredholm modules in the stable Ruelle algebra come from projections in the unstable Ruelle algebra. It appears that the projection used to define the Fredholm module pairs with the fundamental class $\Delta \in KK^1(S \rtimes_{\alpha_s} \mathbb{Z} \otimes U \rtimes_{\alpha_u} \mathbb{Z}, \mathbb{C})$ to give the class of the Fredholm module appearing above. We aim to explore this relationship further in future work.

Chapter 2

Smale Spaces

2.1 Preliminaries

Smale spaces are topological dynamical systems with extra structure. In order to make this work as self contained as possible we will begin with a very brief introduction to topological dynamical systems. For a more detailed account see Lind and Marcus [29] or Brin and Stuck [5].

Let (X, d) be a compact metric space and $\varphi : X \to X$ a homeomorphism. We will denote the corresponding dynamical system by (X, d, φ). Suppose (X, d, φ) and (Y, d', ψ) are both dynamical systems, then we say (X, d, φ) is *conjugate* to (Y, d', ψ) if there is a homeomorphism $\pi : X \to Y$ such that $\pi \circ \varphi = \psi \circ \pi$. Conjugacy is the correct notion of isomorphism for topological dynamical systems. We wish to study properties that are preserved under conjugacy.

In general, dynamical systems (X, d, φ) can be quite unruly and we wish to restrict our attention to dynamical systems with certain recurrence properties. The simplest notion of recurrence is that of a periodic point. We say that x in X is a *periodic point* if $\varphi^n(x) = x$ for some n in $\mathbb{N}$. The least integer n for which this holds is called the *order* of the periodic point. A *fixed point* is a periodic point with order one. Let us denote by

$Per_n(X,\varphi)$ the set of all periodic points with order n and define

$$Per(X,\varphi) = \bigcup_{n\in\mathbb{N}} Per_n(X,\varphi).$$

Observe that $Per(X,\varphi)$ is a φ-invariant subset of X. In the sequel we will be interested in finite subsets of X consisting of orbits of periodic points. The orbit of a point x in X is given by

$$\mathcal{O}(x) = \{\varphi^n(x) | n \in \mathbb{Z}\}.$$

A point x in X is *non-wandering* if for every open set U in X with $x \in U$, there is a positive integer n such that $\varphi^n(U) \cap U$ is non-empty. We shall denote the set of non-wandering points in X by $NW(X,\varphi)$ and observe that this is a closed φ-invariant subset of X, see [35].

There are also notions of recurrence for the whole dynamical system (X,d,φ). We say that (X,d,φ) is *non-wandering* if every point of X is non-wandering. We say that (X,d,φ) is *irreducible* if, for every ordered pair of non-empty open sets U and V in X, there exists a positive integer n such that $\varphi^n(U) \cap V$ is non-empty. Moreover, we say that (X,d,φ) is *mixing* if, for every ordered pair of non-empty open sets U and V, there is a positive integer N such that $\varphi^n(U) \cap V$ is nonempty, for all $n \geq N$.

It is obvious that every mixing dynamical system is irreducible and every irreducible dynamical system is non-wandering. However, the converse of each of these statements is false. Indeed, let X consist of two points and φ the map which exchanges the points, then (X,d,φ) is irreducible but not mixing. Now if we keep X as two points and let φ' be the identity map then (X,d,φ') is non-wandering but not irreducible.

The remainder of this section will be devoted to defining topological entropy for a dynamical system (X,d,φ). Entropy is a measure of the complexity of the mapping φ and is invariant under conjugacy. Entropy is given by the exponential growth rate of the number of essentially different orbit segments of length n. Furthermore, entropy is, in general, the most computable of all invariants of a dynamical system and has far reaching applications. We follow the exposition given in [5].

Let (X, d, φ) be a dynamical system. For each n in $\mathbb{N}$, define a metric that measures the maximum distance between the first n iterates of x and y in X by

$$d_n(x, y) = \sup_{0 \leq k \leq n-1} d(\varphi^k(x), \varphi^k(y)).$$

Fix $\varepsilon > 0$. We say a subset $A \subseteq X$ is (n, ε)-*spanning* if for each x in X there is a y in A such that $d_n(x, y) < \varepsilon$. Since X is compact it follows that there are finite (n, ε)-spanning sets. Define

$$span(n, \varepsilon, \varphi) = \inf\{\#A | A \text{ is an } (n, \varepsilon) - \text{spanning set}\}.$$

Similarly, we say a subset $A \subseteq X$ is (n, ε)-*separated* if for any x and y in A we have $d_n(x, y) > \varepsilon$. Define

$$sep(n, \varepsilon, \varphi) = \sup\{\#A | A \text{ is an } (n, \varepsilon) - \text{separated set}\}$$

which makes sense since every (n, ε)-separated set is finite and compactness gives an upper bound for each n. Now let $\mathcal{B}_\varepsilon$ be the collection of all finite covers of X by sets with d_n diameter less than ε. Let $\#B$ denote the cardinality of each finite cover B in $\mathcal{B}_\varepsilon$. Define

$$cov(n, \varepsilon, \varphi) = \inf\{\#B | B \text{ is in } \mathcal{B}_\varepsilon\}.$$

The *topological entropy* of (X, d, φ) is given by any of the three quantities

$$\begin{aligned} h(X, \varphi) &= \lim_{\varepsilon \to 0} \left(\limsup_{n \to \infty} \frac{1}{n} \log(span(n, \varepsilon, \varphi)) \right) \\ h(X, \varphi) &= \lim_{\varepsilon \to 0} \left(\limsup_{n \to \infty} \frac{1}{n} \log(sep(n, \varepsilon, \varphi)) \right) \\ h(X, \varphi) &= \lim_{\varepsilon \to 0} \left(\limsup_{n \to \infty} \frac{1}{n} \log(cov(n, \varepsilon, \varphi)) \right). \end{aligned}$$

Furthermore, as proved in [5], the limit is in the extended positive real numbers and the limit is independent of the metric generating the topology of X.

2.2 Smale Spaces

In this section, we give an introduction to Smale Spaces. The lecture notes of Ian Putnam [35] were used extensively to produce the exposition here. The reader is also referred to [34, 37, 42] for excellent accounts.

Suppose (X, d) is a compact metric space and $\varphi : X \to X$ is a homeomorphism. We shall begin by giving a heuristic definition of a Smale Space rather than the rigorous one. We say (X, d, φ) is a Smale Space if X is locally a hyperbolic product space with respect to φ; that is, there is a global constant $\varepsilon_X > 0$ such that if x is in X we have two sets $X^s(x, \varepsilon_X)$ and $X^u(x, \varepsilon_X)$ whose intersection is $\{x\}$ and the Cartesian product of these sets is homeomorphic to a neighborhood of x. Moreover, we call $X^s(x, \varepsilon_X)$ the local stable set of x because for any point y on $X^s(x, \varepsilon_X)$ we require that $d(\varphi(x), \varphi(y)) < \lambda^{-1}d(x, y)$ where $\lambda > 1$ is globally defined. Similarly, the local unstable set has the same property if we replace φ with φ^{-1}.

To make this definition rigorous requires us to assume the existence of a map, called the bracket, satisfying certain axioms. The idea of the bracket is to encode the local product structure; if $d(x, y) < \varepsilon_X$, then $[x, y] = \{X^s(x, \varepsilon_X) \cap X^u(y, \varepsilon_X)\}$.

Assume that there is a constant ε_X and a map $[\cdot, \cdot] : \Delta_{\varepsilon_X} \to X$, where

$$\Delta_{\varepsilon_X} = \{(x, y) | d(x, y) < \varepsilon_X\},$$

satisfying the following four bracket axioms:

B1. $[x, x] = x$,

B2. $[x, [y, z]] = [x, z]$ whenever both sides are defined,

B3. $[[x, y], z] = [x, z]$ whenever both sides are defined,

B4. $[\varphi(x), \varphi(y)] = \varphi([x, y])$ whenever both sides are defined.

Moreover, for all x in X and a given global constant $\lambda > 1$ we also have the following contraction axioms:

C1. for y, z such that $d(x, y), d(x, z) \leq \varepsilon_X$ and $[y, x] = x = [z, x]$, we have

$$d(\varphi(y), \varphi(z)) \leq \lambda^{-1} d(y, z),$$

C2. for y, z such that $d(x, y), d(x, z) \leq \varepsilon_X$ and $[x, y] = x = [x, z]$, we have

$$d(\varphi^{-1}(y), \varphi^{-1}(z)) \leq \lambda^{-1} d(y, z).$$

Using the bracket axioms we define the local stable and unstable sets of a point x in X as

$$\begin{aligned} X^s(x, \varepsilon) &= \{y \in X | d(x, y) < \varepsilon \text{ and } [y, x] = x\} \text{ and} \\ X^u(x, \varepsilon) &= \{y \in X | d(x, y) < \varepsilon \text{ and } [x, y] = x\} \end{aligned}$$

where $0 \leq \varepsilon \leq \varepsilon_X$. Figure 2.1 on page 11 illustrates the bracket.

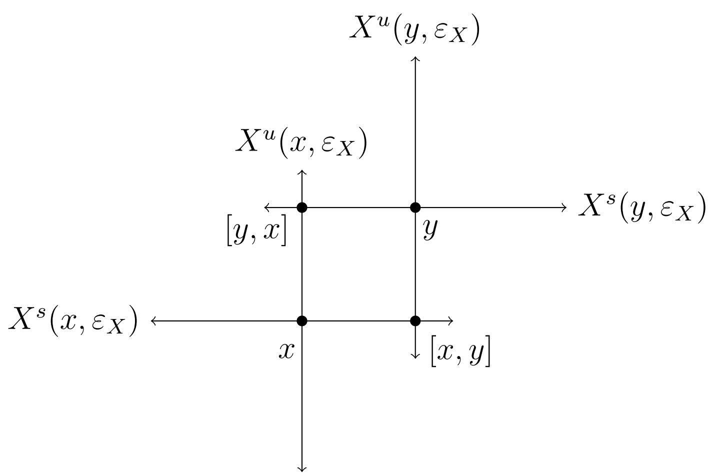


Figure 2.1: The bracket map

2.2.1 Definition. A dynamical system (X, d, φ) having a bracket map satisfying the above axioms is a Smale space.

We note immediately that the bracket map is unique; that is, any map satisfying the above axioms is the bracket map [35]. The following theorem appears in [35] and we quote the theorem and proof here for completeness. We observe that the theorem shows how the bracket gives rise to the local product structure.

2.2.2 Theorem ([35]). *There is $0 \leq \varepsilon'_X \leq \varepsilon_X/2$ such that, for every $0 < \varepsilon < \varepsilon'_X$,*

$$[\cdot,\cdot] : X^u(x,\varepsilon) \times X^s(x,\varepsilon) \to X$$

is a homeomorphism onto its image, which is a neighbourhood of x. We will denote this range by $U(x,\varepsilon)$.

Proof. We begin with the observation that the bracket map is well defined by the triangle inequality. Moreover, since the bracket map is jointly continuous we may find $0 < \delta < \varepsilon_X$ such that, for all x, y with $d(x,y) \leq \delta$, we have $d(x,[x,y]) \leq \varepsilon_X/2$ and $d(x,[y,x]) \leq \varepsilon_X/2$. Now choose $0 < \varepsilon'_X \leq \varepsilon_X/2$ so that for all y, z with $d(x,y) \leq \varepsilon'_X$ and $d(x,z) \leq \varepsilon'_X$, we have $d(x,[y,z]) \leq \delta$. We can define a map η on a neighbourhood of x via $\eta(y) = ([y,x],[x,y])$. By the choice of ε'_X this map is defined on the range of the bracket map. It is also clearly continuous. It is clear from axioms B1, B2, and B3 that the composition $[\cdot,\cdot] \circ \eta$ is the identity. Moreover, if we begin with y in $X^u(x,\varepsilon)$ and z in $X^s(x,\varepsilon)$, then we have

$$\begin{aligned}
\eta([y,z]) &= ([[y,z],x],[x,[y,z]]) && \\
&= ([y,x],[x,z]) && \text{by axioms B2 and B3} \\
&= (y,z) && \text{since } y \in X^u(x,\varepsilon) \text{ and } z \in X^s(x,\varepsilon).
\end{aligned}$$

The conclusion follows. □

2.2.3 Corollary. *There is a constant $0 \leq \varepsilon'_X \leq \varepsilon_X/2$ such that, if $d(x,y) < \varepsilon'_X$, then both $d(x,[x,y]) < \varepsilon_X/2$ and $d(y,[x,y]) < \varepsilon_X/2$ and hence $[x,y]$ is in $X^s(x,\varepsilon_X/2)$ and in $X^u(y,\varepsilon_X/2)$.*

In [35], Putnam goes on to prove a variety of interesting results. We compile a selection of them, without proof, in the following theorem. We note that the final statement says that the bracket map is uniquely determined by (X,d,φ) provided that it exists.

2.2.4 Theorem ([35]). *Suppose that (X,d,φ) is a Smale space. There is a constant $0 < \varepsilon_1 \leq \varepsilon_X$ such that for all $0 < \varepsilon < \varepsilon_1$ the following hold:*

1. *If x and y are in X and $d(\varphi^n(x),\varphi^n(y)) \leq \varepsilon$, for all $n \geq 0$, then y is in $X^s(x,\varepsilon)$.*

2. *If x and y are in X and $d(\varphi^n(x), \varphi^n(y)) \leq \varepsilon$, for all $n \leq 0$, then y is in $X^u(x, \varepsilon)$.*

3. *The map φ is expansive for the constant ε_1; that is, if x and y are in X and $d(\varphi^n(x), \varphi^n(y)) \leq \varepsilon_1$, for all integers n, then $x = y$.*

4. *If x and y are in X and $d(x, y), d(x, [x, y]), d(y, [x, y])$ are all less than $\varepsilon_1/2$, then $[x, y]$ is the unique point in $X^s(x, \varepsilon_1/2) \cap X^u(y, \varepsilon_1/2)$.*

In the previous section we defined nonwandering, irreducible, and mixing dynamical systems. We noted that, for a general dynamical system, mixing implies irreducible and irreducible implies nonwandering but the converses are false. For example, a finite disjoint union of irreducible Smale spaces is nonwandering and not irreducible. The remarkable fact for Smale spaces is that every nonwandering Smale space arises in this way [35, 42]. A similar result holds for irreducible Smale spaces. The following three theorems appear in [35, 42] and the first two are known as *Smale's spectral decomposition.*

2.2.5 Theorem ([35, 42])**.** *Let (X, d, φ) be a nonwandering Smale space. Then there are open, closed, pairwise disjoint, φ-invariant subsets $X_1, \cdots, X_n$ of X, whose union is X, and so that $(X_i, d, \varphi|_{X_i})$ is irreducible, for each $1 \leq i \leq n$. Moreover, these sets are unique up to relabelling.*

2.2.6 Theorem ([35, 42])**.** *Let (X, d, φ) be an irreducible Smale space. Then there are open, closed, pairwise disjoint sets $X_1, \cdots, X_n$ of X, whose union is X. These sets are cyclically permuted by φ and $\varphi^n|_{X_i}$ is mixing for every $1 \leq i \leq n$.*

2.2.7 Theorem ([35, 42])**.** *Let (X, d, φ) be a nonwandering Smale space. Then the periodic points, $Per(X, \varphi)$, are dense in X.*

We now define global stable and unstable equivalence relations on X. Given a point x in X we define the stable and unstable equivalence classes of x by

$$\begin{aligned} X^s(x) &= \{y \in X | \lim_{n \to +\infty} d(\varphi^n(x), \varphi^n(y)) = 0\}, \\ X^u(x) &= \{y \in X | \lim_{n \to +\infty} d(\varphi^{-n}(x), \varphi^{-n}(y)) = 0\}. \end{aligned}$$

We shall also employ the notation $x \sim_s y$ if y is in $X^s(x)$ and $x \sim_u y$ if y is in $X^u(x)$. To see the connection between the global stable and local stable set of a point, we note that, for any x in X and $\varepsilon > 0$, we have $X^s(x, \varepsilon) \subset X^s(x)$. Furthermore, a point y is in $X^s(x)$ if and only if there exists $n \geq 0$ such that $\varphi^n(y)$ is in $X^s(\varphi^n(x), \varepsilon)$. Making the obvious modifications, the same is true in the unstable situation. There is one final equivalence relation on (X, d, φ) called homoclinic equivalence. For a point x in X we define the homoclinic equivalence class of x by

$$X^h(x) = X^s(x) \cap X^u(x),$$

and we also denote homoclinic equivalence by $x \sim_h y$.

As topological spaces the stable and unstable equivalence classes are quite unseemly with respect to the relative topology of X. In fact, if (X, d, φ) is irreducible it follows that both the stable and unstable equivalence classes of a point are dense in X [42]. To rectify this situation we observe that the local stable sets form a neighborhood base for a topology on the global stable sets; that is, given an equivalence class $X^s(x)$, the collection $\{X^s(y, \delta) | y \in X^s(x) \text{ and } \delta > 0\}$ is a neighbourhood base for a Hausdorff and locally compact topology on $X^s(x)$. We define a topology on the unstable equivalence classes in an analogous fashion.

2.3 Examples of Smale Spaces

Examples of Smale spaces include subshifts of finite type, hyperbolic toral automorphisms, solenoids, Smale's horseshoe, and the dynamical system of aperiodic substitution tilings. We present the two extreme cases. A shift of finite type is totally disconnected and has no smooth structure whatsoever. On the other hand, a hyperbolic toral automorphism is smooth and the equivalence classes look like lines.

A much more thorough treatment is given in [35] where solenoids and certain aperiodic substitution tilings are also presented.

2.3.1 Shifts of Finite Type

In this section we introduce one of the fundamental examples of a Smale space. We will not provide any proofs but rather state all the fundamental properties necessary in the sequel. There are many fantastic references to shifts of finite type. Here, we have truncated the presentation from [35] and [28]. However, a general definition is given in [29] and [5], wherein, it is shown that every shift of finite type is topologically conjugate to one given by the following description.

Suppose G is a directed graph with vertex set V and edge set E. There are two maps $i : E \to V$ and $t : E \to V$ where $i(e)$ gives the initial vertex for the directed edge $e \in E$ and $t(e)$ gives the terminal vertex. Given G, we define a compact metric space as follows. Define

$$X_G = \{\cdots e_{-2}e_{-1}.e_0e_1e_2\cdots \mid e_i \in E \text{ for all } i \in \mathbb{Z} \text{ and } t(e_i) = i(e_{i+1})\};$$

that is, X_G consists of all possible bi-infinite paths in the graph G. A metric is defined on X_G, for e and f in X_G, via

$$d(e, f) = \inf\{2^{-n} \mid e_i = f_i \text{ for all } |i| < n\}.$$

We note that it is not hard to see that X_G is compact and totally disconnected. We still require a homeomorphism. Indeed, define φ_G to be the left shift map given by

$$\varphi_G(\cdots e_{-2}e_{-1}.e_0e_1e_2\cdots) = \cdots e_{-2}e_{-1}e_0.e_1e_2\cdots$$

Notice that the sequence has moved to the left by one entry as observed by looking at the placement of the period in each sequence. We note that an element of X_G can also be denoted by $(e_i)_{i\in\mathbb{Z}}$ in which case the shift is given entry-wise by $(\varphi_G(e))_i = e_{i+1}$. We note that φ_G is a homeomorphism on X_G and we therefore have a dynamical system (X_G, φ_G).

Let A be an $N \times N$ matrix with non-negative integer entries. We can construct a directed graph G_A, from A, as follows. Define a vertex set $V = \{v_1, v_2, \cdots v_N\}$ and define $A(i, j)$ edges from vertex v_i to vertex v_j for all $1 \leq i, j \leq N$. We call G_A the graph associated with A. On the other hand, given a directed graph G with vertex set $V = \{v_1, \cdots v_N\}$ we can produce an $N \times N$ matrix A_G by defining entry $A(i, j)$ to be the number of edges from v_i to v_j. We call A_G the adjacency matrix for the directed graph G. From this point forward, we freely interchange between non-negative integer matrices and directed graphs and denote the associated dynamical system by either (X_A, φ_A) or (X_G, φ_G).

Let us give an example of the graph arising from the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}.$$

We depict the graph G_A in Figure 2.2 on page 16. Of course, given the graph, the formulas above also define the integer matrix A_G, via the discussion in the previous paragraph.

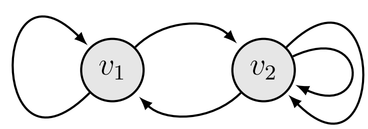


Figure 2.2: The graph associated with the integer matrix A.

We will now show how the system (X_G, φ_G) is a Smale space. Let $\varepsilon_{X_G} = \frac{1}{2}$ and $\lambda = 2$. Suppose $e, f \in X_G$ with $d(e, f) \leq \frac{1}{2}$. Then we define the bracket $[\cdot, \cdot]$ by

$$[e, f]_n = \begin{cases} e_n & \text{if } n \geq 0 \\ f_n & \text{if } n \leq 0. \end{cases}$$

Notice that because $d(e, f) \leq \frac{1}{2}$ it follows that $e_0 = f_0$ so that $[e, f]$ is a well defined path in G and therefore $[e, f] \in X_G$. For a shift of finite type, verification of the bracket axioms is almost trivial, so we have defined a Smale space (X_G, d, φ_G).

Now for the equivalence relations, first let us fix a point e in X_G. Then,

$$\begin{aligned} X^s(e, \varepsilon_{X_G}) &= \{f \in X_G | d(e,f) \leq \varepsilon_{X_G} \text{ and } [e,f] = f\} \\ &= \{f \in X_G | e_i = f_i \text{ for all } i \geq 0\}, \\ X^u(e, \varepsilon_{X_G}) &= \{f \in X_G | d(e,f) \leq \varepsilon_{X_G} \text{ and } [e,f] = e\} \\ &= \{f \in X_G | e_i = f_i \text{ for all } i \leq 0\}. \end{aligned}$$

From the local stable and unstable sets we determine that

$$\begin{aligned} e \sim_s f &\iff \text{there exists } N \in \mathbb{N} \text{ such that } e_n = f_n \text{ for all } n \geq N \\ e \sim_u f &\iff \text{there exists } N \in \mathbb{N} \text{ such that } e_n = f_n \text{ for all } n \leq -N; \end{aligned}$$

that is, paths are stably equivalent if they are right tail equivalent and unstably equivalent if they are left tail equivalent.

Finally, we remark on the topological entropy of an irreducible shift of finite type. We quote Theorem 4.3.1 from [29].

2.3.1 Theorem ([29])**.** *Let (X_A, φ_A) be an irreducible shift of finite type with non-negative integer matrix A. Suppose λ is the Perron-Frobenius eigenvalue for A, then the topological entropy of (X_A, φ_A) is*

$$h(X_A) = \log_2(\lambda).$$

2.3.2 Hyperbolic Toral Automorphisms

We examine a specific example of a hyperbolic toral automorphism, however, the construction is quite general and can be extended, see the discussion at the end of this section. Let A be the matrix

$$A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

and note that $\det(A) = -1$ so that $A(\mathbb{Z}^2) \subset \mathbb{Z}^2$. We may therefore consider A as a map on the quotient space $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$ and to be specific we will denote this map by φ. Let $q : \mathbb{R}^2 \to \mathbb{T}^2$ be the quotient map and put the usual quotient metric on $\mathbb{T}^2$. We claim

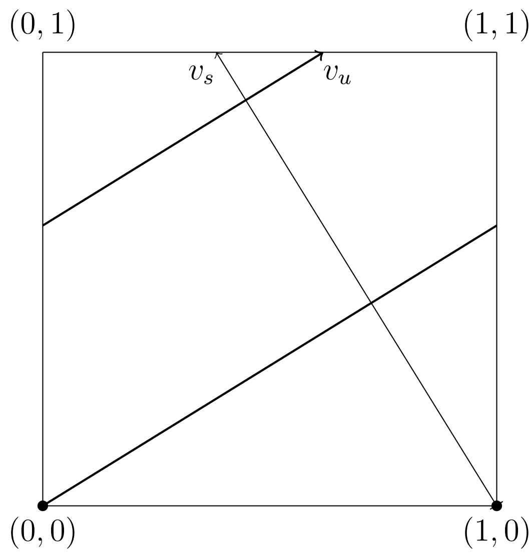


Figure 2.3: Hyperbolic Toral Automorphism

that $(\mathbb{T}^2, d, A)$ is a Smale space.

To see the local product structure we need to describe the eigenvalues and eigenvectors associated with A. Let $\gamma = \frac{1+\sqrt{5}}{2}$ be the golden mean and the eigenvalues for A are γ and $-\gamma^{-1}$. Now the eigenvectors for A are

$$v_s = \begin{pmatrix} 1 \\ -\gamma \end{pmatrix} \qquad \text{and} \qquad v_u = \begin{pmatrix} \gamma \\ 1 \end{pmatrix}$$

where $Av_s = -\gamma^{-1} v_s$ and $Av_u = \gamma v_u$. The situation is illustrated in Figure 2.3 on page 18.

In the usual way the eigenvectors give a basis for a coordinate system. Let $0 < \varepsilon < \frac{1}{2}$ and fix a point $x \in \mathbb{T}^2$, the local stable and unstable equivalence classes are given by

$$\begin{aligned} X^s(x, \varepsilon) &= \{q(x + tv_s) \mid |t| < \varepsilon\}, \\ X^u(x, \varepsilon) &= \{q(x + tv_u) \mid |t| < \varepsilon\}. \end{aligned}$$

Moreover, since $\gamma^{-1} < 1$, for any point $y \in X^s(x, \varepsilon)$ we have $d(\varphi(x), \varphi(y)) < \gamma^{-1} d(x, y)$ and for any point $z \in X^u(x, \varepsilon)$ we have $d(\varphi^{-1}(x), \varphi^{-1}(z)) < \gamma^{-1} d(x, z)$. Finally the global stable and unstable equivalence classes of a point x in $\mathbb{T}^2$ are defined by

$$\begin{aligned} X^s(x) &= \{q(x+tv_s) \mid t \in \mathbb{R}\} \text{ and} \\ X^u(x) &= \{q(x+tv_u) \mid t \in \mathbb{R}\}. \end{aligned}$$

We also note that any $n \times n$ integer matrix B defines a hyperbolic toral automorphism of $\mathbb{T}^n$ if $|\det(B)| = 1$ and the eigenvalues of B do not lie on the unit circle. See [5] for further details.

Chapter 3

C^*-algebras of Smale Spaces

In this chapter we will construct several C^*-algebras from an irreducible Smale space. These C^*-algebras are referred to as the stable, unstable, and homoclinic algebras. Renault's construction of a C^*-algebra from a groupoid is used on the groupoids associated with the equivalence relations defined on Smale spaces. In [34], Putnam constructed C^*-algebras from the stable, unstable, and homoclinic equivalence relations. Putnam and Spielberg refined these constructions in [37] and defined C^*-algebras that are equivalent, in the sense of Muhly, Renault, and Williams [31], to the aforementioned stable, unstable, and homoclinic C^*-algebras but which are étale. We follow the development in [35]. Finally, we construct the stable and unstable Ruelle algebras associated with a Smale space [34, 37].

3.1 Étale groupoids on Smale Spaces

Let (X, d, φ) be a Smale space and let P and Q be finite sets of φ-invariant periodic points. At this point we make no restrictions on P and Q, however, in chapter 4 we will add the assumption that P and Q are disjoint. Define

$$X^s(P) = \bigcup_{p \in P} X^s(p) \quad , \quad X^u(Q) = \bigcup_{q \in Q} X^u(q) \quad , \quad X^h(P, Q) = X^s(P) \cap X^u(Q).$$

3.1.1 Lemma ([42]). *If (X, d, φ) is an irreducible Smale space and P and Q are both φ-invariant sets of periodic points, then $X^h(P, Q)$ is dense in X.*

We now define three groupoids on (X, d, φ) as follows:

$$\begin{aligned} G^s(X, \varphi, Q) &= \{(v, w) | v \sim_s w \text{ and } v, w \in X^u(Q)\} \\ G^u(X, \varphi, P) &= \{(v, w) | v \sim_u w \text{ and } v, w \in X^s(P)\} \\ G^h(X, \varphi) &= \{(v, w) | v \sim_h w\}. \end{aligned}$$

We remark that $G^s(X, \varphi, Q)$ is a closed transversal to stable equivalence on (X, d, φ) and $G^u(X, \varphi, P)$ is a closed transversal unstable equivalence, in the sense of Muhly, Renault, and Williams [31]. In our case, a transversal is a groupoid that intersects every equivalence class. It follows that $G^s(X, \varphi, Q)$ and $G^u(X, \varphi, P)$ are both countable [37]. Moreover, $G^h(X, \varphi)$ is countable by construction [34, 43].

We aim to define an étale topology on these three groupoids. We will restrict our attention to $G^s(X, \varphi, Q)$, since the construction for $G^u(X, \varphi, P)$ is completely analogous. Suppose $v \sim_s w$ and $v, w \in X^u(Q)$. Since $v \sim_s w$ it follows that there exists N such that

$$\varphi^N(w) \in X^s(\varphi^N(v), \varepsilon_X/2).$$

By the continuity of φ, we can define $0 < \delta < \varepsilon_X/2$ so that

$$\begin{aligned} \varphi^n(X^u(v, \delta)) &\subset X^u(\varphi^n(v), \varepsilon_X/2) \qquad \text{for all } 0 \leq n \leq N \text{ and} \\ \varphi^n(X^u(w, \delta)) &\subset X^u(\varphi^n(w), \varepsilon_X/2) \qquad \text{for all } 0 \leq n \leq N. \end{aligned}$$

Given N, δ we define a map h^u on $X^u(w, \delta)$ via

$$h^u(x) = \varphi^{-N}[\varphi^N(x), \varphi^N(v)].$$

The map h^u is illustrated in Figure 3.1 on page 22.

3.1.2 Lemma ([35]). *Let v, w in X be such that $v \sim_s w$ and $v, w \in X^u(Q)$. There exists $0 < \delta \leq \varepsilon_X/2$ and an integer N such that the map $h^u : X^u(w, \delta) \to X^u(v, \delta)$ is a local homeomorphism.*

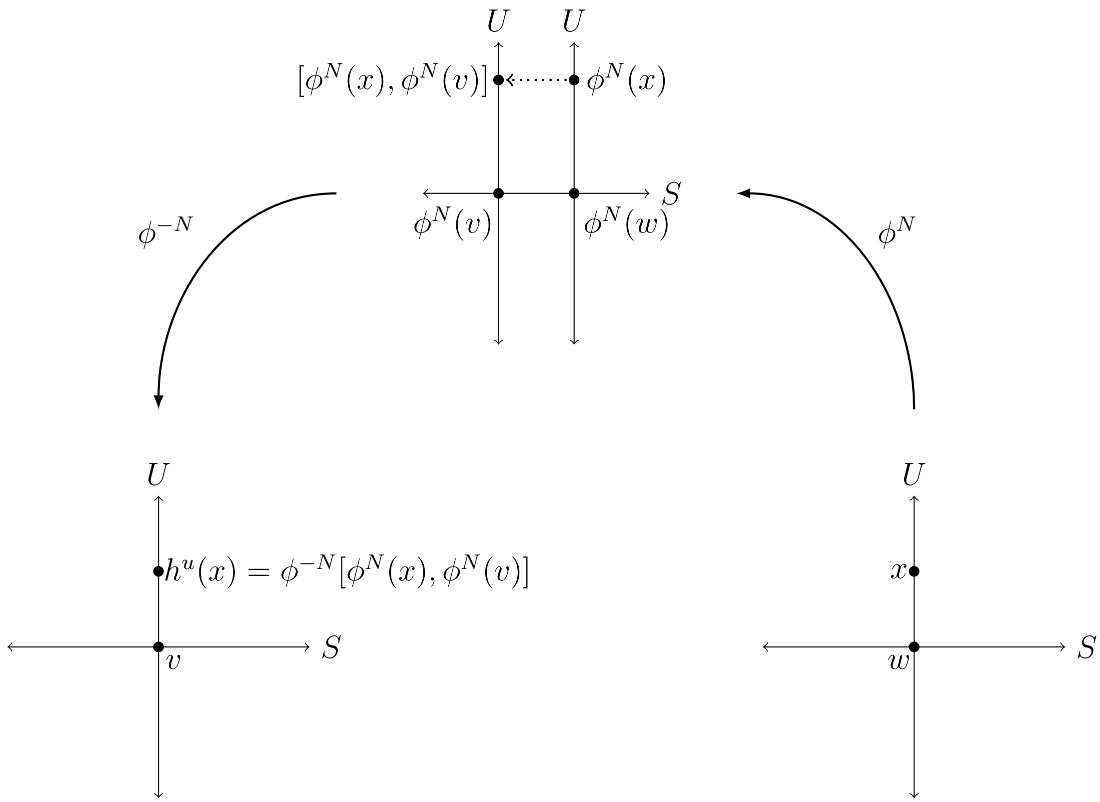


Figure 3.1: The local homeomorphism $h^u : X^u(w, \delta) \to X^u(v, \delta)$

Proof. The existence of δ and N is shown above. We begin by showing that h^u is well defined. Let $x \in X^u(w, \delta)$, since $d(\varphi^N(v), \varphi^N(w)) < \varepsilon_X/2$ and $d(\varphi^N(w), \varphi^N(x)) < \varepsilon_X/2$ it follows that $[\varphi^N(x), \varphi^N(v)]$ is defined by the triangle inequality. Moreover, $[\varphi^N(x), \varphi^N(v)]$ is in both $X^s(\varphi^N(x), \varepsilon_X/2)$ and $X^u(\varphi^N(v), \varepsilon_X/2)$ and it follows that $h^u(x) = \varphi^{-N}[\varphi^N(x), \varphi^N(v)]$ is in both $X^s(x)$ and $X^u(v, \delta)$. Now observe that h^u is a composition of continuous maps and hence is continuous. Furthermore, if we reverse the roles of v and w we obtain another map $g^u : X^u(v, \delta) \to X^u(w, \delta)$. We claim $g^u = (h^u)^{-1}$.

Indeed,

$$\begin{aligned} g^u(h^u(x)) &= g^u(\varphi^N[\varphi^N(x), \varphi^N(v)]) \\ &= \varphi^{-N}[\varphi^N \varphi^{-N}[\varphi^N(x), \varphi^N(v)], \varphi^N(w)] \\ &= \varphi^{-N}[[\varphi^N(x), \varphi^N(v)], \varphi^N(w)] \\ &= \varphi^{-N}(\varphi^N(x)) \\ &= x \end{aligned}$$

where the second last step of the computation follows from the fact that $\varphi^N(x)$ is the unique point in the local stable set of $[\varphi^N(x), \varphi^N(v)]$ and the local unstable set of $\varphi^N(w)$. Similarly, we also have, for y in $X^u(v,\delta)$, that $h^u(g^u(y)) = y$. Finally, it follows from the definition that $h^u(w) = v$. The result follows. □

3.1.3 Theorem ([35])**.** *Let v, w in X be such that $v \sim_s w$ and $v, w \in X^u(Q)$ and let N, δ, h^u be defined by lemma 3.1.2. The collection of sets*

$$V^u(v, w, h^u, \delta) = \{(h^u(x), x) | x \in X^u(w, \delta), h^u(x) \in X^u(v, \delta)\}$$

form a neighbourhood base for a topology on $G^s(X, \varphi, Q)$. In this topology, the range and source maps take each element in the neighbourhood base homeomorphically to an open set in $X^u(Q)$. Moreover, this topology makes $G^s(X, \varphi, Q)$ a second countable, locally compact, Hausdorff groupoid. That is, $G^s(X, \varphi, Q)$ is an étale groupoid.

It is quite clear that we can repeat the above construction for $G^u(X, \varphi, P)$ and obtain the following analogue of theorem 3.1.3.

3.1.4 Theorem ([35])**.** *Let v, w in X be such that $v \sim_u w$ and $v, w \in X^s(P)$ where N, δ, h^s are defined in an analogous fashion with theorem 3.1.3. The collection of sets*

$$V^s(v, w, h^s, \delta) = \{(h^s(x), x) | x \in X^s(w, \delta), h^s(x) \in X^s(v, \delta)\}$$

form a neighbourhood base for a topology on $G^u(X, \varphi, P)$. In this topology, the range and source maps take each element in the neighbourhood base homeomorphically to an open set in $X^s(P)$. Moreover, this topology makes $G^u(X, \varphi, P)$ a second countable, locally compact, Hausdorff groupoid. That is, $G^u(X, \varphi, P)$ is an étale groupoid.

Lastly, we require a topology on $G^h(X,\varphi)$. Recall that $v \sim_h v$ is v and w are both stably and unstably equivalent. Now $G^h(X,\varphi)$ is the set of all such pairs. Now suppose $v \sim_h w$. Then $v \sim_s w$ and from the construction above we obtain N_s, δ_s, and h^u. Similarly, $v \sim_u w$ and we obtain N_u, δ_u, and h^s. Now define $N = \max\{N_s, N_u\}$ and $\delta = \min\{\delta_s, \delta_u\}$. Suppose x is in $B(w,\delta)$, then $[x,w] \in X^u(w,\delta)$ and $[w,x] \in X^s(w,\delta)$. Now $h^u : X^u(w,\delta) \to X^u(v,\delta)$ and $h^s : X^s(w,\delta) \to X^s(v,\delta)$ are local homeomorphisms. Therefore, the map $h : B(w,\delta) \to B(v,\delta)$ defined, for x in $B(w,\delta)$, via

$$h(x) = [h^u([x,w]), h^s([w,x])].$$

is a local homeomorphism. We have the following theorem.

3.1.5 Theorem ([35])**.** *Let v, w in X be such that $v \sim_h w$ where N, δ, h are defined above. The collection of sets*

$$V^h(v,w,h,\delta) = \{(h(x),x) | x \in B(w,\delta), h(x) \in B(v,\delta)\}$$

form a neighbourhood base for a topology on $G^h(X,\varphi)$. In this topology, the range and source maps take each element in the neighbourhood base homeomorphically to an open set in X. Moreover, this topology makes $G^h(X,\varphi)$ a second countable, locally compact, Hausdorff groupoid. That is, $G^h(X,\varphi)$ is an étale groupoid.

3.2 The Stable and Unstable C^*-algebras of a Smale Space

We aim to study groupoid C^*-algebras on the étale groupoids we have constructed on an irreducible Smale space. To accomplish this, we apply Renault's construction [39].

Note that the construction of C^*-algebras from $G^u(X,\varphi,P)$ and $G^h(X,\varphi)$ is completely analogous to the construction for $G^s(X,\varphi,Q)$. We shall outline the construction for $G^s(X,\varphi,Q)$.

We shall denote the continuous functions of compact support on $G^s(X,\varphi,Q)$ by $C_c(G^s(X,\varphi,Q))$, which is a complex linear space. A product and involution are defined on $C_c(G^s(X,\varphi,Q))$ as follows, for $f,g \in C_c(G^s(X,\varphi,Q))$ and $(x,y) \in G^s(X,\varphi,Q)$,

$$\begin{aligned} f \cdot g(x,y) &= \sum_{z \sim_s x} f(x,z)g(z,y) \\ f^*(x,y) &= \overline{f(y,x)}. \end{aligned}$$

This makes $C_c(G^s(X,\varphi,Q))$ into a complex $*$-algebra. Note that it is not obvious that the product is well-defined. We import the result and proof from [35]

3.2.1 Proposition ([35])**.** *Any function in $C_c(G^s(X,\varphi,Q))$ may be written as a sum of functions, each having support in an element of the neighbourhood base described in 3.1.3. Moreover, the product $f \cdot g$ is in $C_c(G^s(X,\varphi,Q))$.*

Proof. Let f be in $C_c(G^s(X,\varphi,Q))$ and let K be the support of f. For each point in K, choose an element of the neighbourhood base that contains the point. These open sets cover K, so by compactness we can choose a finite subcover, say $V^u(v_i,w_i,h^u,\delta_i)$ where $i = 1,2,\cdots,n$ and $0 < \delta \leq \varepsilon_X/2$. Now choose $\delta > 0$ to be smaller than all δ_i and define $\eta : V^u(Q) \to [0,1]$ via

$$\eta(x,y) = \sup\left\{ 0, 1 - (2\delta)^{-1}(d(x,x') + d(y,y')) \;\middle|\; \begin{array}{l} x' \in X^u(x,\delta), y' \in X^u(y,\delta), \\ (x',y') \in K \end{array} \right\}.$$

Now η is a continuous function of compact support on $G^s(X,\varphi,Q)$ such that $\eta|_K = 1$. For each $i = 1,2,\cdots,n$, let

$$\eta_i(x,y) = \begin{cases} \sup\{0, (\delta_i - d(x,v_i))(\delta_i - d(y,w_i)) & \text{if } x \in X^u(v_i,\delta_i), y \in X^u(w_i,\delta_i) \\ 0 & \text{otherwise.} \end{cases}$$

Now η_i is also a continuous function of compact support on $G^s(X,\varphi,Q)$ and $\eta_i > 0$ on $V^u(v_i,w_i,h^u,\delta_i)$. Finally, for each $i = 1,2,\cdots,n$ define

$$f_i(x,y) = \begin{cases} \eta(x,y)f(x,y)\eta_i(x,y)\left(\sum_{j=1}^n \eta_j(x,y)\right)^{-1} & \text{if } \eta(x,y) \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

These functions are all continuous and compactly supported on $G^s(X,\varphi,Q)$. Moreover, $\sum f_i = f$ so that we have proved the first statement. To show that $f \cdot g$ is in

$C_c(G^s(X, \varphi, Q))$ we observe that writing $f = \sum f_i$ and $g = \sum g_i$ makes $f \cdot g$ a finite pointwise product of continuous functions of compact support. □

We aim to define a norm on $C_c(G^s(X, \varphi, Q))$ and then complete $C_c(G^s(X, \varphi, Q))$ in this norm to define a C^*-algebra. At this point there are several options. First we could look at all possible representations of $C_c(G^s(X, \varphi, Q))$ as operators on a Hilbert space. From these Hilbert spaces we obtain a norm and the completion is called the full C^*-algebra. Alternatively, we could consider a single representation on each equivalence class, called the regular representation. This gives rise to a norm called the reduced norm and the completion is called the reduced C^*-algebra. In fact, it is shown in [37] that the groupoid of stable equivalence is amenable so that the full and reduced groupoid C^*-algebras are isomorphic.

3.2.2 Definition. The stable C^*-algebra, $S(X, \varphi, Q)$, is the completion of $C_c(G^s(X, \varphi, Q))$ in the reduced norm.

A third option is possible when (X, d, φ) is irreducible, which is called the fundamental representation [35, 37]. We aim to represent $C_c(G^s(X, \varphi, Q))$ as operators on the Hilbert space

$$\mathcal{H} = \ell^2(X^h(P, Q)).$$

To that end, for $f \in C_c(G^s(X, \varphi, Q))$ and $\xi \in \mathcal{H}$, define the representation $\pi_s : C_c(G^s(X, \varphi, Q)) \to \mathcal{B}(\mathcal{H})$ via

$$\pi_s(f)\xi(x) = \sum_{(x,y) \in G^s(X,\varphi,Q)} f(x, y)\xi(y).$$

With this formula, $\pi_s(f)$ is a bounded linear operator on $\mathcal{H}$. Moreover, we can complete $\pi_s(C_c(G^s(X, \varphi, Q)))$ in the operator norm on this Hilbert space to obtain a C^*-algebra.

Let us comment on the generality of this construction. We recall that in the case that (X, d, φ) is mixing it follows that $X^s(P)$ and $X^u(Q)$ are dense. Therefore, π_s is a faithful representation to the reduced C^*-algebra and hence is isometric. Therefore, the full, reduced and fundamental C^*-algebras of $G^s(X, \varphi, Q)$ are all isomorphic. Furthermore, $S(X, d, \varphi)$ is simple in this case [37]. Now if (X, d, φ) is irreducible then according to theorem 2.2.6 there are N distinct mixing components that are cyclically permuted by

φ so that $X^s(P)$ and $X^u(Q)$ are dense in each component. Therefore, π_s is faithful and $S(X, d, \varphi)$ is a direct sum of N simple components. Finally, if (X, d, φ) is nonwandering then we must adjust our assumptions on the φ-invariant sets of periodic points P and Q. If we assume that both P and Q meet each irreducible component as described in theorem 2.2.5 then π_s is again faithful. We also note that $S(X, \varphi, Q)$ is separable, nuclear, and stable [37, 34].

Similarly, we define the following representations of the unstable and homoclinic groupoids. The unstable groupoid has representation $\pi_u : C_c(G^u(X, \varphi, P)) \to \mathcal{B}(\mathcal{H})$, for $g \in C_c(G^u(X, \varphi, P))$ and $\xi \in \mathcal{H}$, defined by

$$\pi_u(g)\xi(x) = \sum_{(x,y)\in G^u(X,\varphi,P)} g(x, y)\xi(y).$$

The homoclinic representation $\pi_h : C_c(G^h(X, \varphi)) \to \mathcal{B}(\mathcal{H})$, for $h \in C_c(G^h(X, \varphi))$ and $\xi \in \mathcal{H}$, is defined by

$$\pi_h(h)\xi(x) = \sum_{(x,y)\in G^h(X,\varphi)} h(x, y)\xi(y).$$

3.2.3 Definition. The unstable C^*-algebra, $U(X, \varphi, P)$, is the completion of $C_c(G^u(X, \varphi, P))$ in the reduced norm.

3.2.4 Definition. The homoclinic C^*-algebra, $H(X, \varphi)$, is the completion of $C_c(G^h(X, \varphi))$ in the reduced norm.

From proposition 3.2.1, we can write each element of $f \in C_c(G^s(X, \varphi, Q))$ as a finite sum of functions a with support in a neighbourhood base set of the form $V^u(v, w, h^u, \delta)$. We use functions of this form so often in the sequel that we completely describe them in the following lemma, which follows from the definitions.

3.2.5 Lemma. *Suppose a is a function in $C_c(G^s(X, \varphi, Q))$ with support on a basic set $V^u(v, w, h^u, \delta)$ with $v \sim_s w$, $v, w \in X^u(Q)$ and $h^u : X^u(w, \delta) \to X^u(v, \delta)$ a homeomorphism. Then, for $\delta_x \in \mathcal{H}$,*

$$\pi_s(a)\delta_x = \begin{cases} a(h^u(x), x)\delta_{h^u(x)} & \text{if } x \in X^u(w, \delta) \text{ and } h^u(x) \in X^u(v, \delta) \\ 0 & \text{if } x \notin X^u(w, \delta). \end{cases}$$

Define $Source(a) \subseteq X^u(w, \delta)$ to be the points for which a is non-zero on its domain and *define Range(a)* $\subseteq X^u(v,\delta)$ *to be the points in* $X^u(v,\delta)$ *for which* $a(h^u(x),x)\delta_{h^u(x)}$ *is non-zero. Observe that a is zero on the orthogonal complement of* $X^u(w,\delta)$.

Similarly, each element $g \in C_c(G^u(X,\varphi,P))$ can be approximated by as a finite sum of functions b with support in a neighbourhood base set of the form $V^s(v,w,h^s,\delta)$. and each element $h \in C_c(G^h(X,\varphi))$ can be approximated by a finite sum of functions c with support in a neighbourhood base set of the form $V^h(v,w,h,\delta)$. We consider the representation theory of these functions supported on neighbourhood base sets applied to a dirac delta function $\delta_x \in \mathcal{H}$:

$$\pi_u(b)\delta_x = \begin{cases} b(h^s(x),x)\delta_{h^s(x)} & \text{if } x \in X^s(w,\delta) \text{ and } h^s(x) \in X^s(v,\delta) \\ 0 & \text{if } x \notin X^s(w,\delta), \end{cases}$$

$$\pi_h(c)\delta_x = \begin{cases} c(h(x),x)\delta_{h(x)} & \text{if } x \in B(w,\delta) \text{ and } h(x) \in B(v,\delta) \\ 0 & \text{if } x \notin B(w,\delta). \end{cases}$$

We note that every element of any of the above C^*-algebras can be uniformly approximated by a finite sum of functions supported in a neighbourhood base set. We will usually begin by proving theorems by using these functions then appealing to continuity for the general result.

3.3 The Stable and Unstable Ruelle Algebras

The Ruelle algebras, as defined by Putnam in [34], are given by taking the crossed product by the natural actions α_s and α_u on $S(X,\varphi,Q)$ and $U(X,\varphi,P)$ induced by the action φ on X. The Ruelle algebras were shown to be separable, simple, stable, nuclear, and purely infinite when (X,φ) is mixing [37]. Moreover, according to the purely infinite case of Elliott's classification program, as developed by Kirchberg and Phillips, they are completely classified by their K-theory groups. We now embark on their construction.

We will work with the stable C^*-algebra $S(X,\varphi,Q)$ and recall that this algebra has a representation as bounded operators on the Hilbert space $\mathcal{H} = \ell^2(H(P,Q))$. The homeomorphism φ preserves the equivalence relations we are considering and $\varphi \times \varphi$ is an automorphism of the groupoid $G^s(X,\varphi,Q)$, see [37] for an excellent account. Therefore, φ induces an automorphism on the C^*-algebra $S(X,\varphi,Q)$ by

$$\alpha_s(a)(x,y) = a(\varphi^{-1}(x),\varphi^{-1}(y))$$

where a is in $S(X,\varphi,Q)$ and (x,y) are in $G^s(X,\varphi,Q)$. The homeomorphism φ also induces a canonical unitary on the Hilbert space $\mathcal{H}$ via

$$u\delta_x = \delta_{\varphi(x)}.$$

3.3.1 Lemma. *The pair (π_s, u) are covariant for $S(X,\varphi,Q)$; that is, $\pi_s(\alpha_s(a)) = u\pi_s(a)u^*$ for all a in $S(X,\varphi,Q)$.*

Proof. Let δ_x be a basis element in $\mathcal{H}$ and let a in $S(X,\varphi,Q)$ be supported on a basic set of the form $V^u(v,w,h^u,\delta)$. We begin by computing

$$\begin{aligned}
\pi_s(\alpha_s(a))\delta_x(y) &= \sum_{(y,z)\in G^s(X,\varphi,Q)} \alpha_s(a)(y,z)\delta_x(z) \\
&= \alpha_s(a)(y,x) \\
&= \begin{cases} a(\varphi^{-1}(y),\varphi^{-1}(x)) & \text{if } \varphi^{-1}(x)\in X^u(w,\delta) \text{ and } \varphi^{-1}(y) = h^u\circ\varphi^{-1}(x) \\ 0 & \text{otherwise} \end{cases} \\
&= a(h^u\circ\varphi^{-1}(x),\varphi^{-1}(x))\delta_{\varphi\circ h^u\circ\varphi^{-1}(x)}.
\end{aligned}$$

So we have,

$$\begin{aligned}
u\pi_s(a)u^*\delta_x &= u\pi_s(a)\delta_{\varphi^{-1}(x)} \\
&= ua(h^u\circ\varphi^{-1}(x),\varphi^{-1}(x))\delta_{h^u\circ\varphi^{-1}(x)} \\
&= a(h^u\circ\varphi^{-1}(x),\varphi^{-1}(x))\delta_{\varphi\circ h^u\circ\varphi^{-1}(x)} \\
&= \pi_s(\alpha_s(a))\delta_x,
\end{aligned}$$

and covariance is proved in this case. The general case follows from continuity. □

3.3.2 Definition ([34])**.** The stable Ruelle algebra, denoted by $S\rtimes_{\alpha_s}\mathbb{Z}$, is defined as the crossed product:

$$S(X,\varphi,Q)\rtimes_{\alpha_s}\mathbb{Z}.$$

For the unstable algebra, we note that, for b in $U(X, \varphi, P)$ and (x, y) in $G^u(X, \varphi, P)$, we have $\alpha_u(b)(x, y) = b(\varphi^{-1}(x), \varphi^{-1}(y))$. We can apply the analogous construction of the crossed product. We have the following.

3.3.3 Definition ([34])**.** The unstable Ruelle algebra, denoted by $U \rtimes_{\alpha_u} \mathbb{Z}$, is defined as the crossed product:

$$U(X, \varphi, P) \rtimes_{\alpha_u} \mathbb{Z}.$$

Chapter 4

Poincaré Duality for Smale Spaces

4.1 Kasparov's KK-theory

For a Smale space (X, d, φ) and φ-invariant sets of periodic points P and Q in X, we have constructed C^*-algebras $S(X, \varphi, Q)$ and $U(X, \varphi, P)$. These C^*-algebras have natural integer actions, α_s and α_u, coming from the original homeomorphism φ, which give rise to the stable and unstable Ruelle algebras, $S \rtimes_{\alpha_s} \mathbb{Z}$ and $U \rtimes_{\alpha_u} \mathbb{Z}$. We also note that all four of these C^*-algebras are independent of our choice of P and Q, up to Morita equivalence [37].

The duality theorem for Smale Spaces relates the K-theory of the stable Ruelle algebra, $S \rtimes_{\alpha_s} \mathbb{Z}$, with the K-homology of the unstable Ruelle algebra, $U \rtimes_{\alpha_u} \mathbb{Z}$. In a similar manner, the K-theory of the unstable Ruelle algebra is related to the K-homology of the stable Ruelle algebra.

For our purposes, it is convenient to work with Kasparov's KK-theory where K-theory and K-homology can be defined simultaneously:

$$\begin{aligned} KK^*(A, \mathbb{C}) &\cong K^*(A) = K - \text{homology} \\ KK^*(\mathbb{C}, A) &\cong K_*(A) = K - \text{theory.} \end{aligned}$$

In this section we will briefly outline the notation and results from KK-theory used in the sequel. Our account of KK-theory will be rather heuristic but captures the ideas required in this chapter. All C^*-algebras we consider will be ungraded which will simplify the theory. See [25] for an introduction to Kasparov theory. For more details see any of [3, 16, 26] and the references therein.

Let A and B be ungraded C^*-algebras. Then cycles of $KK(A, B)$ are given by pairs $(\mathcal{E}, F)$ where $\mathcal{E}$ is an $A - B$ Hilbert bimodule, and F is an adjointable operator on $\mathcal{E}$ satisfying, for all a in A,

$$a(F^*F - 1) \in \mathcal{K}(\mathcal{E}) \quad , \quad a(FF^* - 1) \in \mathcal{K}(\mathcal{E}) \quad , \quad [a, F] \in \mathcal{K}(\mathcal{E}).$$

Elements of $KK(A, B)$ can be thought of as generalized morphisms from A to B with a product given as composition of morphisms, this idea can be made precise. In fact, KK-theory is an additive category with pairs of C^*-algebras as objects and morphisms from A to B as elements of $KK(A, B)$. Moreover, KK is a functor from C^*-algebras to $\mathbb{Z}/2\mathbb{Z}$-graded abelian groups. The abelian group associated with a pair of C^*-algebras (A, B) is denoted $KK(A, B)$, and is contravariant in the first variable and covariant in the second. For further details see [23].

Let us show specifically how a $*$-homomorphism between C^*-algebras gives rise to a class in KK-theory.

4.1.1 Example. Suppose that we have a $*$-homomorphism $\phi : A \to B$. Then, ϕ defines an element of $KK(A, B)$ in the following way. In the standard manner, let $\mathcal{E} = B$ be the $B - B$ Hilbert bi-module with inner product given by $< x, y >_B = x^*y$ for $x, y \in B$. Observe that $\mathcal{E}$ is a left B module since B acts as adjointable operators by left multiplication. Therefore, $\mathcal{E}$ is also an $A - B$ module via the $*$-homomorphism ϕ, for $a \in A$ and $e \in \mathcal{E}$ we have $a \cdot e := \varphi(a)e$. Define F to be the zero adjointable operator acting on $\mathcal{E}$ by left multiplication. We claim that $(\mathcal{E}, F)$ determines a class in $KK(A, B)$. We must show that, for all a in A,

$$\phi(a)(F^*F - 1) \in \mathcal{K}(\mathcal{E}) \quad , \quad \phi(a)(FF^* - 1) \in \mathcal{K}(\mathcal{E}) \quad , \quad [\phi(a), F] \in \mathcal{K}(\mathcal{E}).$$

However, these all follow trivially provided that B acts as compact operators on $\mathcal{E}$ as a left module since we would then have $\phi : A \to \mathcal{K}(\mathcal{E})$. Indeed, let b be in B and L_b the operator of left multiplication on $\mathcal{E}$ via $L_b(x) = bx$. Now $L : b \to L_b$ is an isomorphism of B onto a closed $*$-subalgebra of the adjointable operators on $\mathcal{E}$. Since

$$\Theta_{b,x}(y) = b < x, y >_B = bx^*y = L_{bx^*}(y),$$

it follows that the closure of linear spans of products in B under L is $\mathcal{K}(\mathcal{E})$, since $\Theta_{b,x}$ is a rank one adjointable operator. See example 2.26 in [38] for more details. Therefore, $\phi : A \to B$ defines an element of $KK(A, B)$.

In the sequel, the KK-class given by the identity will be used extensively. Let A be a C^*-algebra and $1 : A \to A$ be the identity automorphism. Let $1_A \in KK(A, A)$ denote the class given by the above construction.

Perhaps the most important aspect of KK-theory is the existence of the Kasparov product. We first introduce the intersection (cap) product and then the cap-cup product after introducing some notation. There is a bilinear pairing: $KK(A, D) \otimes_D KK(D, B) \to KK(A, B)$, called the Kasparov intersection product [25]. The definition of the intersection product is, in general, quite complicated. However, if $\alpha \in KK(A, D)$ and $\beta \in KK(D, B)$ have representations as $*$-homomorphisms, then the cap product is given by composition; that is, $\alpha \otimes_D \beta$ is the map

$$A \xrightarrow{\ \alpha\ } D \xrightarrow{\ \beta\ } B.$$

In the sequel we shall adopt the notation appearing in [26]. Denote $C_0(0, 1)$ by $\mathscr{S}$. Now $KK^1(A, B)$ is, by definition, $KK(A \otimes \mathscr{S}, B)$. If A and B are separable and A is nuclear then it follows that $KK^1(A, B) \cong Ext(A, B)$ [25]. Let $A_1, A_2, \cdots, A_n$ and $B_1, B_2, \cdots, B_n$ be C^*-algebras and denote by σ_{ij} and σ^{ij} the homomorphisms, induced by the obvious isomorphisms on the tensor products of C^*-algebras,

$$\begin{aligned}\sigma_{ij} : KK(A_1 \otimes \cdots \otimes A_i \otimes \cdots \otimes A_j \otimes \cdots \otimes A_n, B)& \\ \to KK(A_1 \otimes \cdots \otimes A_j \otimes \cdots \otimes A_i \otimes \cdots \otimes A_n, B)&, \\ \sigma^{ij} : KK(A, B_1 \otimes \cdots \otimes B_i \otimes \cdots \otimes B_j \otimes \cdots \otimes B_n)& \\ \to KK(A, B_1 \otimes \cdots \otimes B_j \otimes \cdots \otimes B_i \otimes \cdots \otimes B_n)&.\end{aligned}$$

Let $\tau_D : KK^i(A,B) \to KK^i(A\otimes D, B\otimes D)$ be the natural map $x \mapsto x\otimes 1_D$ and $\tau^D : KK^i(A,B) \to KK^i(D\otimes A, D\otimes B)$ be the natural map $x \mapsto 1_D \otimes x$. We will use $\tau_D(x)$ and $x\otimes 1_D$ interchangeably as warranted by clarity of notation, similarly for $\tau^D(x)$ and $1_D\otimes x$.

We are now in a position to define Kasparov's cap-cup product [25]. Suppose $x_1 \in KK(A_1, B_1\otimes D)$ and $x_2 \in KK(D\otimes A_2, B_2)$. Then the product $x_1 \otimes_D x_2 : KK(A_1, B_1\otimes D)\otimes KK(D\otimes A_2, B_2) \to KK(A_1\otimes A_2, B_1\otimes B_2)$ is defined by

$$x_1 \otimes_D x_2 = (x_1 \otimes 1_{A_2}) \otimes_{B_1\otimes D\otimes A_2} (1_{B_1}\otimes x_2).$$

Notice that for $B_1 = \mathbb{C} = A_2$ we obtain the usual cap product. Moreover, the cap-cup product is the cap product of $x_1\otimes 1_{A_2}$ and $1_{B_1}\otimes x_2$.

Let $\mathcal{T}$ be the Toeplitz algebra, which is defined as the C^*-algebra generated by the unilateral shift operator and the identity operator on $\ell^2(\mathbb{N})$. Define $z : [0,1] \to S^1$ via $z(t) = e^{2\pi i t}$ and observe that z generates $C(S^1)$ as a C^*-algebra. Now let $\mathcal{T}$ also denote the Toeplitz extension

$$0 \longrightarrow \mathcal{K}(\ell^2(\mathbb{N})) \longrightarrow \mathcal{T} \longrightarrow C(S^1) \longrightarrow 0$$

which is an element of $KK^1(C(S^1),\mathbb{C})$. Observe that $z-1$ generates $\mathscr{S} \subset C(\mathbb{T})$ and we denote the corresponding restriction of the Toeplitz extension by $\mathcal{T}_0$, which is an element of $KK(\mathscr{S}\otimes\mathscr{S},\mathbb{C})$. Now if β in $KK(\mathbb{C},\mathscr{S}\otimes\mathscr{S})$ is the Bott element, see 19.2.5 in [3], then we have

$$\beta \otimes_{\mathscr{S}\otimes\mathscr{S}} \mathcal{T}_0 = 1_{\mathbb{C}} \quad \text{and} \quad \mathcal{T}_0 \otimes \beta = 1_{\mathscr{S}\otimes\mathscr{S}},$$

see Section 19.2 in [3].

We shall also require conditions under which the Kasparov product commutes.

4.1.2 Lemma ([25] p.159)**.** *If x is in $KK^i(A_1,B_1)$ and y is in $KK^j(A_2,B_2)$, then*

$$\begin{aligned}&(x\otimes 1_{A_2}) \otimes_{B_1\otimes A_2} (1_{B_1}\otimes y)\\ &\quad = (-1)^{ij}\sigma_{12}\sigma^{12}((y\otimes 1_{A_1})\otimes_{B_2\otimes A_1}(1_{B_2}\otimes x)) \in KK(A_1\otimes A_2, B_1\otimes B_2).\end{aligned}$$

The following lemma follows from the definitions. In fact, if one views an extension as a $*$-homomorphism from a C^*-algebra into the Calkin algebra then the product in the lemma is just a composition of $*$-homomorphisms and hence gives another extension. See Lemma 2.6.1 in [24] for the isomorphism between extensions and $*$-homomorphisms into the Calkin Algebra.

4.1.3 Lemma. *Suppose $\phi : A \to D$ is a $*$-homomorphism and a class y in $KK^1(D, B)$ is represented by an extension*

$$0 \longrightarrow B \otimes \mathcal{K}(\mathcal{H}) \longrightarrow \mathcal{E} \xrightarrow{\ \pi\ } D \longrightarrow 0.$$

Then, the intersection product $x \otimes_D y \in KK^1(A, B)$ is represented by an extension

$$0 \longrightarrow B \otimes \mathcal{K}(\mathcal{H}) \longrightarrow \mathcal{E}' \xrightarrow{\ \pi'\ } A \longrightarrow 0$$

where $\mathcal{E}'$ is the pull-back $\mathcal{E}' = \{x \in \mathcal{E} \mid \pi'(x) = \phi^{-1}(\pi(x))\}$.

4.2 Poincaré Duality

In this section, we present a definition of Poincaré duality appropriate to the C^*-algebras we wish to study. We note that the definition given here is the odd version of the definition given by Kasparov [25] and Connes [8].

4.2.1 Definition. Let A and B be C^*-algebras. Suppose we have two classes Δ in $KK^1(A \otimes B, \mathbb{C})$ and δ in $KK^1(\mathbb{C}, A \otimes B)$. We say that A and B are *Poincaré dual* if

$$\delta \otimes_B \Delta = 1_A \quad \text{and} \tag{4.1}$$

$$\delta \otimes_A \Delta = -1_B. \tag{4.2}$$

Notation. In the previous definition and for the remainder of this dissertation we shall employ the following notation:

$$\begin{aligned} \delta \otimes_B \Delta &= \sigma_{12}(\delta \otimes_B \sigma_{12}(\Delta)) \in KK(A \otimes \mathscr{S} \otimes \mathscr{S}, A) \quad \text{and} \\ \delta \otimes_A \Delta &= \sigma_{12}(\sigma^{12}(\delta) \otimes_A \Delta) \in KK(B \otimes \mathscr{S} \otimes \mathscr{S}, B). \end{aligned}$$

Moreover, we also note that the above formulas have Bott periodicity encoded into the definition. That is, without employing Bott periodicity we assume

$$\begin{aligned} \delta \otimes_B \Delta &= \tau^A(\mathcal{T}_0) \quad \text{and} \\ \delta \otimes_A \Delta &= -\tau^B(\mathcal{T}_0). \end{aligned}$$

From the definition of Poincaré duality and the Kasparov product we obtain isomorphisms between the K-theory of A and the K-homology of B. The remainder of this section is dedicated to explaining these isomorphisms. We note that we are forced to bring Bott periodicity directly into the definition since our duality is odd.

4.2.2 Definition ([26])**.** Let A and B be C^*-algebras. Suppose we have two classes Δ in $KK^1(A \otimes B, \mathbb{C})$ and δ in $KK^1(\mathbb{C}, A \otimes B)$. We obtain homomorphisms $\Delta_i : K_i(A) \to K^{i+1}(B)$ and $\delta_i : K^i(B) \to K_{i+1}(A)$ via

$$\begin{aligned} \Delta_0(x) &= x \otimes_A \Delta & x &\in K_0(A), \\ \Delta_1(x) &= \beta \otimes_{\mathscr{S} \otimes \mathscr{S}} (\sigma_{12}(x \otimes_A \Delta)) & x &\in K_1(A), \\ \delta_1(y) &= \beta \otimes_{\mathscr{S} \otimes \mathscr{S}} (\delta \otimes_B y) & y &\in K^1(B), \\ \delta_0(y) &= \delta \otimes_B y & y &\in K^0(B). \end{aligned}$$

4.2.3 Theorem ([18])**.** *Let A and B be C^*-algebras. Suppose we have two classes Δ in $KK^1(A \otimes B, \mathbb{C})$ and δ in $KK^1(\mathbb{C}, A \otimes B)$ that implement Poincaré duality between A and B. Then,*

$$\begin{aligned} \delta_{i+1} \circ \Delta_i &= (-1)^i 1_{K_i(A)} \\ \Delta_{i+1} \circ \delta_i &= (-1)^{i+1} 1_{K^i(B)} \end{aligned}$$

where $i+1$ is interpreted mod(2). Moreover, we obtain isomorphisms $K_i(A) \cong K^{i+1}(B)$.

This theorem has been proven in [18], however, for completeness we give the proof here as well. To accomplish this we use results from [26]. Given a class in either K-theory or K-homology, the idea is to uncouple the class from the product given in the above composition. Then, using Poincaré duality, we obtain the identity. Once we have

$$\begin{aligned}\delta_{i+1} \circ \Delta_i &= (-1)^i 1_{K_i(A)} \\ \Delta_{i+1} \circ \delta_i &= (-1)^{i+1} 1_{K^i(B)}\end{aligned}$$

the final statement in the theorem follows from an algebra argument. The uncoupling steps are given in lemmas 4.2.4 and 4.2.5.

4.2.4 Lemma ([26])**.** *Let A and B be C^*-algebras. Suppose we have classes Δ in $KK^1(A \otimes B, \mathbb{C})$ and δ in $KK^1(\mathbb{C}, A \otimes B)$. Then, for x in $K_0(A) = KK(\mathbb{C}, A)$ and y in $K_1(A) = KK(\mathscr{S}, A)$, we have*

$$\begin{aligned}\delta_1 \circ \Delta_0(x) &= x \otimes_A (\beta \otimes_{\mathscr{S} \otimes \mathscr{S}} (\delta \otimes_B \Delta) \quad \textit{and} \\ \delta_0 \circ \Delta_1(y) &= -y \otimes_A (\beta \otimes_{\mathscr{S} \otimes \mathscr{S}} (\delta \otimes_B \Delta).\end{aligned}$$

Proof. From definition 4.2.2,

$$\delta_1 \circ \Delta_0(x) = \beta \otimes_{\mathscr{S} \otimes \mathscr{S}} (\delta \otimes_B (x \otimes_A \Delta)).$$

Consider $Z = \delta \otimes_B (x \otimes_A \Delta)$. Expanding the product we have

$$Z = (\delta \otimes 1_{\mathscr{S}}) \otimes_{A \otimes B \otimes \mathscr{S}} (1_A \otimes x \otimes 1_B \otimes 1_{\mathscr{S}}) \otimes_{A \otimes A \otimes B \otimes \mathscr{S}} (1_A \otimes \Delta). \tag{4.3}$$

Lemma 4.1.2 states that $(1_A \otimes x \otimes 1_B \otimes 1_{\mathscr{S}}) \otimes_{A \otimes A \otimes B \otimes \mathscr{S}} (1_A \otimes \Delta)$ is the same as $(1_A \otimes 1_B \otimes x \otimes 1_{\mathscr{S}}) \otimes_{A \otimes B \otimes A \otimes \mathscr{S}} (1_A \otimes \sigma_{12}(\Delta))$. Putting this back into (4.3) we have

$$Z = (\delta \otimes 1_{\mathscr{S}}) \otimes_{A \otimes B \otimes \mathscr{S}} (1_A \otimes 1_B \otimes x \otimes 1_{\mathscr{S}}) \otimes_{A \otimes B \otimes A \otimes \mathscr{S}} (1_A \otimes \sigma_{12}(\Delta)). \tag{4.4}$$

Now, consider $(\delta \otimes 1_{\mathscr{S}}) \otimes_{A \otimes B \otimes \mathscr{S}} (1_A \otimes 1_B \otimes x \otimes 1_{\mathscr{S}})$ and compute, using lemma 4.1.2,

$$\begin{aligned}(\delta \otimes 1_{\mathscr{S}}) \otimes_{A \otimes B \otimes \mathscr{S}} (1_A \otimes 1_B \otimes x \otimes 1_{\mathscr{S}}) &= ((\delta \otimes 1_{\mathbb{C}}) \otimes_{A \otimes B} (1_{A \otimes B} \otimes x)) \otimes 1_{\mathscr{S}} \\ &= \sigma_{12}\sigma^{12}((x \otimes 1_{\mathscr{S}}) \otimes_{A \otimes \mathscr{S}} (1_A \otimes \delta)) \otimes 1_{\mathscr{S}} \\ &= (x \otimes 1_{\mathscr{S}} \otimes 1_{\mathscr{S}}) \otimes_{A \otimes \mathscr{S} \otimes \mathscr{S}} \sigma^{23}\sigma^{12}(1_A \otimes \delta \otimes 1_{\mathscr{S}}) \\ &= (x \otimes 1_{\mathscr{S}} \otimes 1_{\mathscr{S}}) \otimes_{A \otimes \mathscr{S} \otimes \mathscr{S}} \sigma_{12}(\delta \otimes 1_A \otimes 1_{\mathscr{S}}).\end{aligned}$$

Putting this back into (4.4) we obtain

$$Z = (x \otimes 1_{\mathscr{S}} \otimes 1_{\mathscr{S}}) \otimes_{A\otimes\mathscr{S}\otimes\mathscr{S}} \sigma_{12}(\delta \otimes 1_A \otimes 1_{\mathscr{S}}) \otimes_{A\otimes B\otimes A\otimes\mathscr{S}} (1_A \otimes \sigma_{12}(\Delta)). \tag{4.5}$$

Putting β back into the product we have

$$\delta_1 \circ \Delta_0(x) = \beta \otimes_{\mathscr{S}\otimes\mathscr{S}} (x \otimes 1_{\mathscr{S}} \otimes 1_{\mathscr{S}}) \otimes_{A\otimes\mathscr{S}\otimes\mathscr{S}} \sigma_{12}(\delta \otimes 1_A \otimes 1_{\mathscr{S}}) \otimes_{A\otimes B\otimes A\otimes\mathscr{S}} (1_A \otimes \sigma_{12}(\Delta)).$$

Again using lemma 4.1.2 we see that

$$\beta \otimes_{\mathscr{S}\otimes\mathscr{S}} (x \otimes 1_{\mathscr{S}} \otimes 1_{\mathscr{S}}) = x \otimes_A (1_A \otimes \beta).$$

Furthermore, the term $\sigma_{12}(\delta \otimes 1_A \otimes 1_{\mathscr{S}}) \otimes_{A\otimes B\otimes A\otimes\mathscr{S}} (1_A \otimes \sigma_{12}(\Delta))$ is exactly what we defined as $\delta \otimes_B \Delta$. So simplifying we obtain

$$\delta_1 \circ \Delta_0(x) = x \otimes_A (\beta \otimes_{\mathscr{S}\otimes\mathscr{S}} (\delta \otimes_B \Delta)).$$

Now for y in $K_1(A)$, by definition we have

$$\delta_0 \circ \Delta_1(y) = \delta \otimes_B (\beta \otimes_{\mathscr{S}\otimes\mathscr{S}} (\sigma_{12}(y \otimes_A \Delta))).$$

Expanding the product we have

$$\delta_0 \circ \Delta_1(y) \delta \otimes_{A\otimes B} (1_{A\otimes B} \otimes \beta) \otimes_{A\otimes B\otimes\mathscr{S}\otimes\mathscr{S}} \sigma_{23}(1_A \otimes y \otimes 1_B \otimes 1_{\mathscr{S}}) \otimes_{A\otimes A\otimes B\otimes\mathscr{S}} (1_A \otimes \Delta). \tag{4.6}$$

Using lemma 4.1.2 we have that

$$\delta \otimes_{A\otimes B} (1_{A\otimes B} \otimes \beta) = (\beta \otimes 1_{\mathscr{S}}) \otimes_{\mathscr{S}\otimes\mathscr{S}\otimes\mathscr{S}} (\delta \otimes 1_{\mathscr{S}\otimes\mathscr{S}}).$$

Putting this back into (4.6) we have $\delta_0 \circ \Delta_1(y) =$

$$(\beta \otimes 1_{\mathscr{S}}) \otimes_{\mathscr{S}\otimes\mathscr{S}\otimes\mathscr{S}} (\delta \otimes 1_{\mathscr{S}\otimes\mathscr{S}}) \otimes_{A\otimes B\otimes\mathscr{S}\otimes\mathscr{S}} \sigma_{23}(1_A \otimes y \otimes 1_B \otimes 1_{\mathscr{S}}) \otimes_{A\otimes A\otimes B\otimes\mathscr{S}} (1_A \otimes \Delta).$$

Now observe that up to tensoring by $\mathscr{S}$ and the degree of y, this equation is the same as the previous computation. Therefore, using the previous computation and noting that when we commute y and δ we pick up a negative sign using lemma 4.1.2, we have

$$\delta_0 \circ \Delta_1(y) = -y \otimes_A (\beta \otimes_{\mathscr{S}\otimes\mathscr{S}} (\delta \otimes_B \Delta)).$$

□

4.2.5 Lemma ([26])**.** *Let A and B be C^*-algebras. Suppose we have classes Δ in $KK^1(A \otimes B, \mathbb{C})$ and δ in $KK^1(\mathbb{C}, A \otimes B)$. Then, for x in $K^0(B) = KK(B, \mathbb{C})$ and y in $K^1(B) = KK(B \otimes \mathscr{S}, \mathbb{C})$, we have*

$$\begin{aligned} \Delta_1 \circ \delta_0(x) &= (\beta \otimes_{\mathscr{S}\otimes\mathscr{S}} (\delta \otimes_A \Delta)) \otimes_B x \quad \text{and} \\ \Delta_0 \circ \delta_1(y) &= -(\beta \otimes_{\mathscr{S}\otimes\mathscr{S}} (\delta \otimes_A \Delta)) \otimes_B y. \end{aligned}$$

Proof. From definition 4.2.2,

$$\Delta_1 \circ \delta_0(x) = \beta \otimes_{\mathscr{S}\otimes\mathscr{S}} (\sigma_{12}((\delta \otimes_B x) \otimes_A \Delta))$$

Expansion of the product yields $\Delta_1 \circ \delta_0(x) =$

$$(1_B \otimes \beta) \otimes_{B\otimes\mathscr{S}\otimes\mathscr{S}} \sigma_{12}(\delta \otimes 1_B \otimes 1_{\mathscr{S}}) \otimes_{A\otimes B\otimes B\otimes\mathscr{S}} (1_A \otimes x \otimes 1_B \otimes 1_{\mathscr{S}}) \otimes_{A\otimes B\otimes\mathscr{S}} \Delta. \quad (4.7)$$

Consider the term $(1_A \otimes x \otimes 1_B \otimes 1_{\mathscr{S}}) \otimes_{A\otimes B\otimes\mathscr{S}} \Delta$ and compute, using lemma 4.1.2,

$$\begin{aligned} (1_A \otimes x \otimes 1_B \otimes 1_{\mathscr{S}}) \otimes_{A\otimes B\otimes\mathscr{S}} \Delta &= \sigma_{12}(x \otimes 1_A \otimes 1_B \otimes 1_{\mathscr{S}}) \otimes_{A\otimes B\otimes\mathscr{S}} \Delta \\ &= \sigma_{12}(x \otimes 1_{A\otimes B\otimes\mathscr{S}}) \otimes_{A\otimes B\otimes\mathscr{S}} (1_{\mathbb{C}} \otimes \Delta) \\ &= \sigma_{12}\sigma_{12}\sigma_{23}\sigma_{34}(\Delta \otimes 1_B) \otimes_B x \\ &= \sigma_{12}(1_B \otimes \Delta) \otimes_B x \end{aligned}$$

Putting this back into (4.7) we have

$$\Delta_1 \circ \delta_0(x) = (1_B \otimes \beta) \otimes_{B\otimes\mathscr{S}\otimes\mathscr{S}} \sigma_{12}(\delta \otimes 1_B \otimes 1_{\mathscr{S}}) \otimes_{A\otimes B\otimes B\otimes\mathscr{S}} \sigma_{12}(1_B \otimes \Delta) \otimes_B x. \quad (4.8)$$

Now observe that

$$\sigma_{12}(\delta \otimes 1_B \otimes 1_{\mathscr{S}}) \otimes_{A\otimes B\otimes B\otimes\mathscr{S}} \sigma_{12}(1_B \otimes \Delta) = \sigma_{12}(\sigma^{12}(\delta \otimes 1_B \otimes 1_{\mathscr{S}}) \otimes_{B\otimes A\otimes B\otimes\mathscr{S}} (1_B \otimes \Delta)$$

and, moreover, the latter expression is exactly our definition of $\delta \otimes_A \Delta$. Putting this back into (4.8) and simplifying gives

$$\Delta_1 \circ \delta_0(x) = (\beta \otimes_{\mathscr{S}\otimes\mathscr{S}} (\delta \otimes_A \Delta)) \otimes_B x. \tag{4.9}$$

The proof that

$$\Delta_0 \circ \delta_1(y) = -(\beta \otimes_{\mathscr{S}\otimes\mathscr{S}} (\delta \otimes_A \Delta)) \otimes_B y$$

is completely analogous and is omitted. □

This accomplishes the uncoupling step. Now suppose that A and B are Poincaré dual in the sense of definition 4.2.1. Putting this into the uncoupling lemmas gives the following maps

$$\begin{aligned} \delta_1 \circ \Delta_0(x) &= \quad x \otimes 1_A = x && (x \in K_0(A)) \\ \delta_0 \circ \Delta_1(y) &= -(y \otimes 1_A) = -y && (y \in K_0(A)) \\ \Delta_1 \circ \delta_0(x) &= \quad -1_B \otimes x = -x && (x \in K^0(A)) \\ \Delta_0 \circ \delta_1(y) &= -(-1_B \otimes y) = y && (y \in K^1(A)) \end{aligned}$$

A simple algebra argument shows that these give rise to isomorphisms between $K_i(A)$ and $K^{i+1}(B)$ for $i = 0, 1$. We have therefore proven theorem 4.2.3.

Similarly, we can apply the KK-isomorphisms $\sigma_{12}(\Delta)$ and $\sigma^{12}(\delta)$ to obtain a result analogous with A and B reversed. Note that we must also update our definitions of Δ_* and δ_*.

4.2.6 Theorem ([18]). *Let A and B be C^*-algebras. Suppose we have two classes Δ in $KK^1(A \otimes B, \mathbb{C})$ and δ in $KK^1(\mathbb{C}, A \otimes B)$ that implement Poincaré duality between A and B. Then,*

$$\begin{aligned} \delta_{i+1} \circ \Delta_i &= (-1)^i 1_{K_i(B)} \\ \Delta_{i+1} \circ \delta_i &= (-1)^{i+1} 1_{K^i(A)} \end{aligned}$$

Moreover, we obtain isomorphisms $K_i(B) \cong K^{i+1}(A)$.

4.3 Examples of Poincaré Duality

Alain Connes was the first to exhibit a non-commutative C^*-algebra with Poincaré duality [8]. He managed to show that the irrational rotation algebra A_θ is Poincaré dual to its opposite algebra A_θ^{op}. The duality is even and the definition of duality in the even case can be found in [8]. Details on gradings and opposite algebras can be found in [24].

The second example of non-commutative C^*-algebras having Poincaré duality was discovered by Kaminker and Putnam [26]. In fact, Kaminker and Putnam showed that the stable and unstable Ruelle algebras associated with a shift of finite type were Poincaré dual. To be more precise, Kaminker and Putnam showed that these algebras exhibit Spanier-Whitehead duality, which follows from Poincaré duality and, in fact, their result can be extended. This was the first example of an odd duality. We shall explain Kaminker and Putnam's work in more detail after the introduction to this section.

Heath Emerson [16, 17, 18] showed that a large class of C^*-algebras have odd Poincaré duality. Namely, Emerson exhibited Poincaré duality for groupoid C^*-algebras associated to the groupoids $\partial\Gamma \rtimes \Gamma$ where Γ is a hyperbolic group satisfying certain hypotheses, and $\partial\Gamma$ is its Gromov boundary.

Of course the duality theorem appearing in this dissertation contains Kaminker and Putnam's duality theorem as a special case. It would be remiss not to give a detailed explanation of their original result. Moreover, Kaminker and Putnam have an unpublished manuscript containing an E-theory proof of the general duality theorem for an irreducible Smale space [27]. We note that many of the arguments appearing in their unpublished manuscript are used in this dissertation and we shall point out these occurrences as they appear.

Let (X_A, φ_A) be a shift of finite type associated with a non-negative $N \times N$ integer matrix A which is irreducible. Let A^t denote the transpose of A. In this case, it is known that $S(X_A, \varphi_A) \rtimes_{\alpha_s} \mathbb{Z} \cong \mathcal{O}_{A^t} \otimes \mathcal{K}(\mathcal{H})$ and $U(X_A, \varphi_A) \rtimes_{\alpha_u} \mathbb{Z} \cong \mathcal{O}_A \otimes \mathcal{K}(\mathcal{H})$, where $\mathcal{O}_A$ denotes the Cuntz-Kreiger algebra associated with the matrix A. Therefore, it is appropriate to work with the Cuntz-Kreiger Algebras $\mathcal{O}_A$ and $\mathcal{O}_{A^t}$ since Kasparov theory is stable. Based on work of D. Evans [19] and D. Voiculescu [45], an extension of $\mathcal{O}_A \otimes \mathcal{O}_{A^t}$ can be made using the creation and annihilation operators acting on a subspace, determined by the matrix A, of the full Fock space of a finite dimensional Hilbert space:

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_A \otimes \mathcal{O}_{A^t} \longrightarrow 0.$$

This extension determines the fundamental class Δ in $KK^1(\mathcal{O}_A \otimes \mathcal{O}_{A^t}, \mathbb{C})$. On the other hand, one can take the element

$$w = \sum_{i=1}^{n} s_i^* \otimes t_i \in \mathcal{O}_A \otimes \mathcal{O}_{A^t}$$

and show that $w^*w = ww^*$. From this it follows that there is a map, sending z to w, from $C(S^1)$ into $\mathcal{O}_A \otimes \mathcal{O}_{A^t}$ and the element δ in $KK^1(\mathbb{C}, \mathcal{O}_A \otimes \mathcal{O}_{A^t})$ is determined by its restriction to $\mathscr{S}$. Finally, Kaminker and Putnam [26] proved a criterion for Spanier - Whitehead duality and showed that it is appropriate to the duality classes they constructed, the final step in showing the duality theorem for a shift of finite type.

Let us explicitly write down the K-theory and K-homology groups for a shift of finite type associated with an $N \times N$ irreducible matrix. The C^*-algebras $S(X_A, \varphi_A, Q)$ and $U(X_A, \varphi_A, P)$ are both AF-algebras [35] whose K_0-groups are the inductive limits of

$$\mathbb{Z}^N \xrightarrow{\quad A^t \quad} \mathbb{Z}^N \xrightarrow{\quad A^t \quad} \mathbb{Z}^N \xrightarrow{\quad A^t \quad} \cdots$$

and

$$\mathbb{Z}^N \xrightarrow{\quad A \quad} \mathbb{Z}^N \xrightarrow{\quad A \quad} \mathbb{Z}^N \xrightarrow{\quad A \quad} \cdots$$

Moreover, the K-theory and K-homology groups of $\mathcal{O}_A$ were computed by Cuntz and Krieger [12, 13]

$$\begin{aligned} K_0(\mathcal{O}_A) &\cong \mathbb{Z}^N/(I - A^t)\mathbb{Z}^N, \\ K_1(\mathcal{O}_A) &\cong \ker((I - A^t) : \mathbb{Z}^N \to \mathbb{Z}^N), \\ K^0(\mathcal{O}_A) &\cong \ker((I - A) : \mathbb{Z}^N \to \mathbb{Z}^N), \\ K^1(\mathcal{O}_A) &\cong \mathbb{Z}^N/(I - A)\mathbb{Z}^N. \end{aligned}$$

As in [26], we note that $\mathbb{Z}^N/(I-A)\mathbb{Z}^N \cong \mathbb{Z}^N/(I-A^t)\mathbb{Z}^N$ by the structure theorem for finitely generated abelian groups, but the isomorphism is not natural. The explanation for the isomorphism now comes from the Duality Theorem for shifts of finite type. We also note that there is a C^*-algebra isomorphism between $\mathcal{O}_A$ and $\mathcal{O}_{A^t}$, however, this result requires the Kirchberg - Phillips classification of purely infinite, nuclear, simple C^*-algebras.

In an unpublished manuscript, Kaminker and Putnam went on to prove the duality theorem for all irreducible Smale Spaces [27]. From the beginning, it was clear that a different approach was needed. The first observation was to note that the actions on the stable and unstable C^*-algebras could be extended to continuous $\mathbb{R}$ actions by replacing the crossed products with mapping cylinders. We note that the mapping cylinder of the stable algebra is

$$\begin{aligned} &C(S(X,\varphi,Q),\alpha_s) \\ &\qquad = \{f:\mathbb{R}\to S(X,\varphi,Q) | f \text{ is continuous }, f(t+1)=\alpha_s(f(t)), t\in\mathbb{R}\} \end{aligned}$$

and for $r\in\mathbb{R}$ we have the formula $(\alpha_s)_r(f)(t)=f(t+r)$ which defines an action of $\mathbb{R}$ on $C(S(X,\varphi,Q),\alpha_s)$. We have similar formulas for the unstable algebra and the unstable mapping cylinder $C(U(X,\varphi,P),\alpha_u)$. The crossed product $C(S(X,\varphi,Q),\alpha_s)\rtimes_{\alpha_s}\mathbb{R}$ is KK-equivalent to $C(S(X,\varphi,Q),\alpha_s)$ via the Connes-Thom isomorphism and the crossed product $C(S(X,\varphi,Q),\alpha_s)\rtimes_{\alpha_s}\mathbb{R}$ is Morita equivalent to the Ruelle algebra $S\rtimes_{\alpha_s}\mathbb{Z}$. Now an asymptotic morphism gives a class Δ in $E(C(S(X,\varphi,Q),\alpha_s)\otimes C(U(X,\varphi,P),\alpha_u),\mathbb{C})$ where E denotes Connes-Higson E-theory. Note that KK-theory and E-theory agree for nuclear C^*-algebras. To construct a representative of the class δ in $E(\mathbb{C},C(S(X,\varphi,Q),\alpha_s)\otimes C(U(X,\varphi,P),\alpha_u))$, Putnam had shown in [34] that the product of the stable and unstable equivalence relations are equivalent, in the sense of Muhly, Renault, and Williams [31], to the homoclinic equivalence relation. Therefore, it follows that $S(X,\varphi,Q)\otimes U(X,\varphi,P)$ and the homoclinic algebra $H(X,\varphi)$ are Morita equivalent. Now $H(X,\varphi)$ is unital so one can find the class of the identity in $E(\mathbb{C},S(X,\varphi,Q)\otimes U(X,\varphi,P))$ and extend this to the mapping cylinders. Finally, Kaminker and Putnam produced a very technical argument to prove the duality.

The approach to Poincaré duality presented here has similarities to the E-theory approach. However, we will use KK-theory and the Ruelle algebras themselves rather than mapping cylinders and the crossed products associated with them. Moreover, to construct the fundamental class Δ we introduce a new Hilbert space and exploit the interactions between the stable and unstable Ruelle algebras when represented on the same Hilbert space. The class δ is constructed in much the same manner as in the E-theory approach. We also note that our constructions are very geometrical in nature while the classes from Kaminker and Putnam's original work [26] are combinatorial. We note that the combinatorial nature of shifts of finite type versus the geometrical nature of more general Smale spaces is quite typical.

4.4 Poincaré Duality for Irreducible Smale Spaces

The remainder of the chapter is dedicated to showing that the stable and unstable Ruelle algebras associated with an irreducible Smale space are Poincaré dual.

Recall that (X, d, φ) is an irreducible Smale space and P and Q are φ-invariant sets of periodic points. We now add the additional assumption that P and Q are distinct, which will be used in the following way. Consider $X^h(P, Q)$, the set of points in X which are stably equivalent to a point in P and unstably equivalent to a point in Q. Notice that the only periodic points in $X^s(P)$ are the points in P themselves. For if there was another periodic point then it must be in P since we have assumed φ-invariance. Similarly, the only periodic points in $X^u(Q)$ are the points in Q itself. Now since $X^h(P, Q) = X^s(P) \cap X^u(Q)$ and the fact that $P \cap Q = \emptyset$, there are no periodic points at all in $X^h(P, Q)$. Moreover, up to Morita equivalence, the C^*-algebras $S(X, \varphi, Q)$ and $U(X, \varphi, P)$ are independent of the choice of P and Q.

We now state the main result of the chapter.

4.4.1 Duality Theorem. *Let (X, φ) be an irreducible Smale Space with P and Q finite, φ-invariant sets of periodic points such that $P \cap Q = \emptyset$. Then $S \rtimes_{\alpha_s} \mathbb{Z}$ and $U \rtimes_{\alpha_u} \mathbb{Z}$ are Poincaré dual; that is, there are classes δ in $KK^1(\mathbb{C}, S \rtimes_{\alpha_s} \mathbb{Z} \otimes U \rtimes_{\alpha_u} \mathbb{Z})$ and Δ in*

$KK^1(S \rtimes_{\alpha_s} \mathbb{Z} \otimes U \rtimes_{\alpha_u} \mathbb{Z}, \mathbb{C})$ *such that*

$$\begin{aligned} \delta \otimes_{U \rtimes_{\alpha_u} \mathbb{Z}} \Delta &\cong 1_{S \rtimes_{\alpha_s} \mathbb{Z}} \quad \textit{and} \\ \delta \otimes_{S \rtimes_{\alpha_s} \mathbb{Z}} \Delta &\cong -1_{U \rtimes_{\alpha_u} \mathbb{Z}}. \end{aligned}$$

From the duality theorem, we obtain isomorphisms between the K-theory and K-homology groups of the stable and unstable Ruelle algebras and vice versa.

The remainder of this chapter is organized as follows. In the next two sections, we construct the classes implementing Poincaré duality for the Ruelle algebras. The class δ in $KK^1(\mathbb{C}, S \rtimes_{\alpha_s} \mathbb{Z} \otimes U \rtimes_{\alpha_u} \mathbb{Z})$ is given by defining a $*$-homomorphism from $\mathscr{S}$ to $S \rtimes_{\alpha_s} \mathbb{Z} \otimes U \rtimes_{\alpha_u} \mathbb{Z}$. The class Δ in $KK^1(S \rtimes_{\alpha_s} \mathbb{Z} \otimes U \rtimes_{\alpha_u} \mathbb{Z}, \mathbb{C})$ is given by defining an extension of $S \rtimes_{\alpha_s} \mathbb{Z} \otimes U \rtimes_{\alpha_u} \mathbb{Z}$ using an infinite direct sum of the Hilbert space $\mathcal{H} = \ell^2(X^h(P,Q))$. The final section is the proof of the duality theorem.

4.4.1 The First Duality Class

We give a description of the duality class δ in $KK(\mathscr{S}, S \rtimes_{\alpha_s} \mathbb{Z} \otimes U \rtimes_{\alpha_u} \mathbb{Z})$ for the Ruelle algebras.

Before we begin with the technical details, let us explain the underlying idea of the construction. Consider the product groupoid $G^s(X,\varphi,Q) \times G^u(X,\varphi,P)$ which is equivalent to the groupoid $G^h(X,\varphi)$ in the sense of Muhly, Renault, and Williams [31]. Since the groupoid $G^h(X,\varphi)$ is an étale groupoid with compact unit space, namely X itself, its groupoid C^*-algebra, $H(X,\varphi)$, is unital. Thus, $K_0(H(X,\varphi))$ has a canonical element determined by the class of the identity. The above equivalence of groupoids implies that $S(X,\varphi,Q) \otimes U(X,\varphi,P)$ is Morita equivalent to $H(X,\varphi)$ and we construct a projection in $S(X,\varphi,Q) \otimes U(X,\varphi,P)$ corresponding to the class of the identity in $H(X,\varphi)$. For details regarding the Morita equivalence above see [34] and for a general reference [38].

4.4.2 Definition. Suppose that $\mathcal{F} = \{f_1, f_2, \ldots, f_K\}$ are continuous, non-negative functions on X and $G = \{g_1, \cdots, g_K\}$ is a subset of $X^h(P,Q) = X^s(P) \cap X^u(Q)$. For

$0 < \epsilon \leq \varepsilon'_X$, we say that $(\mathcal{F}, G)$ is an ε-partition of X if

1. the squares of the functions in $\mathcal{F}$ form a partition of unity in $C(X)$; that is,

$$\sum_{k=1}^{K} f_k^2 = 1,$$

2. the elements of G are all distinct,

3. the support of f_k is contained in $B(g_k, \varepsilon/2)$, for each $1 \leq k \leq K$.

4.4.3 Lemma ([27])**.** *There exists $(\mathcal{F}, G)$, an ε'_X-partition of X such that*

$$(\mathcal{F} \circ \varphi^{-1}, \varphi(G)) = (\{f_k \circ \varphi^{-1} \mid 1 \leq k \leq K\}, \{\varphi(g_k) \mid 1 \leq k \leq K\})$$

is also an ε'_X-partition of X. Moreover, G can be chosen so that $G \cap \varphi(G) = \varnothing$.

Proof. Choose $\varepsilon'_X > \varepsilon' > 0$ small enough that, for any x in X, $\varphi(B(x, \varepsilon'/2)) \subseteq B(\varphi(x), \varepsilon'_X/2)$. Let $U_x = B(x, \varepsilon'/4)$ so that $\{U_x\}_{x \in X}$ covers X. Since X is compact there is a finite subcover, say $\{U_k\}_{k=1}^{K}$. Now a partition of unity subordinate to $\{U_k\}_{k=1}^{K}$ exists [4] and we define $\mathcal{F} = \{f_1, f_2, \cdots, f_K\}$ to be the square roots of these functions. By lemma 3.1.1, $X^h(P, Q)$ is dense. So we may choose points g_k in $X^h(P, Q)$ to be within $\varepsilon'/4$ from the center of each ball U_k. Now the support of each function in $\mathcal{F}$ is still contained in a ball of radius $\varepsilon'/2$. Therefore, we have an ε'_X-partition $(\mathcal{F}, G)$ such that $(\mathcal{F} \circ \varphi^{-1}, \varphi(G))$ is also an ε'_X-partition. □

We remark that we do not discount the possibility that some of the functions in $\mathcal{F}$ may be zero for our convenience.

Now, for $0 < \varepsilon \leq \varepsilon'_X$, let $(\mathcal{F}, G)$ be an ε-partition and define a function $p_{\mathcal{F},G}$ on $G^s(X, \varphi, Q) \times G^u(X, \varphi, P)$ by setting

$$p_{\mathcal{F},G}((x, x'), (y, y')) = f_i([x, y]) f_j([x', y']),$$

for $(x,x') \in G^s(X,\varphi,Q), (y,y') \in G^u(X,\varphi,P)$, if, for some i,j,

$$x \in X^u(g_i,\varepsilon), y \in X^s(g_i,\varepsilon), x' \in X^u(g_j,\varepsilon), y' \in X^s(g_j,\varepsilon), [x,y]=[x',y']$$

and to be zero otherwise. Notice that if a pair i,j exist for a given $((x,x'),(y,y'))$, then it is unique, since $g_i=[y,x]$ and $g_j=[y',x']$.

4.4.4 Lemma ([27])**.** *Let $0<\varepsilon\leq\varepsilon'_X$ and let $(\mathcal{F},G)$ be an ϵ-partition. Then $p_{\mathcal{F},G}$ is in $S(X,\varphi,Q)\otimes U(X,\varphi,P)$.*

Proof. Let us fix a pair i,j and suppose there exists $(x,x')\in G^s(X,\varphi,Q)$ and $(y,y')\in G^u(X,\varphi,P)$ such that

$$x \in X^u(g_i,\varepsilon), y \in X^s(g_i,\varepsilon), x' \in X^u(g_j,\varepsilon), y' \in X^s(g_j,\varepsilon), [x,y]=[x',y'].$$

We note that $[g_i,g_j]$ is defined and is stably equivalent to g_i and unstably equivalent to g_j. By lemma 3.1.2 there are local homeomorphisms $h^u: X^u(g_i,\varepsilon)\to X^u([g_i,g_j],\varepsilon)$ and $h^s: X^s(g_j,\varepsilon)\to X^s([g_i,g_j],\varepsilon)$ defined by

$$\begin{aligned} h^u(x) &= [x,[g_i,g_j]] = [x,g_j] \\ h^s(y') &= [[g_i,g_j],y'] = [g_i,y'] \end{aligned}$$

It is immediate that if we let $x'=h^u(x)$ and $y=h^s(y')$ then the points satisfy the conditions above. On the other hand, if $((x,x'),(y,y'))$ satisfy the conditions then we have

$$\begin{aligned} x' &= [[x',y'],x'] = [[x,y],x'] = [x,x'] = [x,g_j] = h^u(x) \\ y &= [y,[x,y]] = [y,[x',y']] = [y,y'] = [g_i,y'] = h^s(y'). \end{aligned}$$

This shows that points satisfying the conditions are realized by local homeomorphisms, one on the local unstable set of g_i and one on the local stable set of g_j.

Set $\varepsilon'>0$. Consider the function on $X^u(g_i,\varepsilon)\times X^s(g_j,\varepsilon)$ which sends (x,y') to $f_i([x,y])f_j([x',y'])$. It is clearly a continuous function of compact support so that it can be uniformly approximated within ε' by a function of the form

$$\sum_{k=1}^{K_{i,j}} a_{i,j,k}(x,x')b_{i,j,k}(y,y')$$

where, for each fixed k, we have $a_{i,j,k}$ in $C_c(G^s(X,\varphi,Q))$ and $b_{i,j,k}$ in $C_c(G^u(X,\varphi,P))$. If there exists no $((x,x'),(y,y'))$ for a fixed i,j we define the above sum to be zero. Now it follows that

$$\sum_{i,j}\sum_{k=1}^{K_{i,j}} a_{i,j,k}\otimes b_{i,j,k}$$

is within ε' of $p_{\mathcal{F},G}$ in norm. This completes the proof. □

In the sequel, it will be convenient to have a description of the operator $p_{\mathcal{F},G}$ on the Hilbert space $\ell^2(X^h(P,Q))\otimes\ell^2(X^h(P,Q))$, in terms of our usual basis, $\{\delta_w\otimes\delta_z \mid w,z\in X^h(P,Q)\}$. We also introduce a standard convention that the bracket map returns the empty set when the bracket of two points is undefined. Of course, any operator applied to the dirac delta function of the empty set will return zero and we declare that any function of the empty set is also zero. This convention will simplify many of the upcoming formulations.

4.4.5 Lemma ([27]). *Let $0<\varepsilon\le\varepsilon'_X$ and let $(\mathcal{F},G)$ be an ε-partition. Suppose w,z are in $X^h(P,Q)$, then we have*

$$p_{\mathcal{F},G}\delta_w\otimes\delta_z = f_k([w,z])\sum_{i=1}^{K} f_i([w,z])\delta_{[w,g_i]}\otimes\delta_{[g_i,z]}$$

if there exists a $1\le k\le K$, such that $w\in X^u(g_k,\varepsilon), z\in X^s(g_k,\varepsilon)$ and is zero if there is no such k. (If the k exists, it is unique, for given w,z. The expression on the right makes sense using our standard convention.)

Proof. For any x, y in $X^h(P,Q)$, we compute

$$\begin{aligned}(p_{\mathcal{F},G}\delta_w \otimes \delta_z)(x,y) &= \sum_{x' \in X^h(x)} \sum_{y' \in X^h(y)} p_{\mathcal{F},G}((x,x'),(y,y'))\delta_w(x')\delta_z(y') \\ &= p_{\mathcal{F},G}((x,w),(y,z)) \\ &= f_i([x,y])f_k([w,z]),\end{aligned}$$

provided

$$x \in X^u(g_i,\varepsilon), y \in X^s(g_i,\varepsilon), w \in X^u(g_k,\varepsilon), z \in X^s(g_k,\varepsilon), [x,y] = [w,z]$$

and zero otherwise. If there is no k such that $w \in X^u(g_k,\varepsilon), z \in X^s(g_k,\varepsilon)$, then the conclusion holds. Let us continue under the assumption that there is such a k (which must be unique, since $[z,w] = g_k$ and the bracket map is (locally) unique in a Smale space). If, for some i, $[w,z]$ is not in the support of f_i, then for any x, y as above for which $[x,y] = [w,z]$, we have $f_i([x,y]) = f_i([w,z]) = 0$. On the other hand, if $[w,z]$ is in the support of f_i, for some i, then

$$\begin{aligned} x &= [x,g_i] = [[x,y],g_i] = [[w,z],g_i] = [w,g_i] \\ y &= [g_i,y] = [g_i,[x,y]] = [g_i,[w,z]] = [g_i,z]. \end{aligned}$$

That is, for a given i, the choice of x, y is unique. For each such i, we have

$$(p_{\mathcal{F},G}\delta_w \otimes \delta_z)([w,g_i],[g_i,z]) = f_i([w,z])f_k([w,z]),$$

and the left hand side is zero for all other values of x, y. The conclusion follows. $\square$

4.4.6 Lemma ([27])**.** *Let $0 < \varepsilon \leq \varepsilon'_X$. If $(\mathcal{F},G)$ is an ε-partition, then $p_{\mathcal{F},G}$ is a projection. If $(\mathcal{F} \circ \varphi^{-1}, \varphi(G))$ is also an ε-partition, then*

$$(u \otimes u)p_{\mathcal{F},G}(u^* \otimes u^*) = p_{\mathcal{F}\circ\varphi^{-1},\varphi(G)}.$$

Proof. To show that $p_{\mathcal{F},G}$ is a projection we use lemma 4.4.5 to compute $p^2_{\mathcal{F},G}\delta_w \otimes \delta_z$.

First of all, we have

$$p_{\mathcal{F},G}\delta_w \otimes \delta_z = f_k([w,z]) \sum_{i=1}^{K} f_i([w,z])\delta_{[w,g_i]} \otimes \delta_{[g_i,z]},$$

if $w \in X^u(g_k,\varepsilon), z \in X^s(g_k,\varepsilon)$ and zero otherwise. We apply $p_{\mathcal{F},G}$ again, taking it through the sum and looking at each term individually. That is, for fixed $1 \leq i \leq K$, we must consider, for what l is $[w,g_i]$ in $X^u(g_l,\varepsilon)$ and $[g_i,z]$ in $X^s(g_l,\varepsilon)$. Since $[w,g_i]$ is clearly in $X^u(g_i)$, this can only happen for $l = i$. Using this, we obtain

$$\begin{aligned}
p^2_{\mathcal{F},G}\delta_w \otimes \delta_z &= f_k([w,z]) \sum_{i=1}^{K} f_i([w,z]) p_{\mathcal{F},G}\delta_{[w,g_i]} \otimes \delta_{[g_i,z]} \\
&= f_k([w,z]) \sum_{i=1}^{K} f_i([w,z]) f_i([w,z]) \sum_{j=1}^{K} f_j([[w,g_i],[g_i,z]])\delta_{[[w,g_i],g_j]} \otimes \delta_{[g_j,[g_i,z]]} \\
&= f_k([w,z]) \sum_{i=1}^{K} f_i([w,z])^2 \sum_{j=1}^{K} f_j([w,z])\delta_{[w,g_j]} \otimes \delta_{[g_j,z]} \\
&= f_k([w,z]) \sum_{j=1}^{K} f_j([w,z])\delta_{[w,g_j]} \otimes \delta_{[g_j,z]} \\
&= p_{\mathcal{F},G}\delta_w \otimes \delta_z
\end{aligned}$$

The second part of the proof is a computation:

$$\begin{aligned}
(u \otimes u)p_{\mathcal{F},G}(u^* \otimes u^*)\delta_w \otimes \delta_z &= (u \otimes u)p_{\mathcal{F},G}\delta_{\varphi^{-1}(w)} \otimes \delta_{\varphi^{-1}(z)} \\
&= (u \otimes u) f_k([\varphi^{-1}(w), \varphi^{-1}(z)]) \sum_{i=1}^{K} f_i([\varphi^{-1}(w), \varphi^{-1}(z)])\delta_{[\varphi^{-1}(w),g_i]} \otimes \delta_{[g_i,\varphi^{-1}(z)]} \\
&= f_k(\varphi^{-1}([w,z])) \sum_{i=1}^{K} f_i(\varphi^{-1}([w,z]))\delta_{\varphi([\varphi^{-1}(w),g_i])} \otimes \delta_{\varphi([g_i,\varphi^{-1}(z)])} \\
&= p_{\mathcal{F}\circ\varphi^{-1},\varphi(G)}.
\end{aligned}$$

□

From Lemma 4.4.3, we may find $\mathcal{F} = \{f_1,\ldots,f_K\}, G = \{g_1,\ldots,g_K\}$ such that $(\mathcal{F},G)$ and $(\mathcal{F}\circ\varphi^{-1},\varphi(G))$ are both ε'_X-partitions of X with $G \cap \varphi(G) = \varnothing$. Since $X^h(P,Q)$

contains no periodic points we know that neither does G. By lemmas 4.4.4 and 4.4.6, we have that both $p_{\mathcal{F},G}$ and $p_{\mathcal{F}\circ\varphi^{-1},\varphi(G)}$ are projections in $S(X,\varphi,Q)\otimes U(X,\varphi,P)$. For each $0\leq s\leq 1$, consider the collection

$$\mathcal{F}_s=\{(1-s)^{1/2}f_1,\ldots,(1-s)^{1/2}f_K,s^{1/2}f_1\circ\varphi^{-1},\ldots,s^{1/2}f_K\circ\varphi^{-1}\}$$

together with the set of points

$$G_s=\{g_1,\ldots,g_K,\varphi(g_1),\ldots,\varphi(g_K)\}$$

Clearly, $(\mathcal{F}_s,G_s)$ is an ε'_X-partition, for all $0\leq s\leq 1$. For convenience, define $p_s=p_{\mathcal{F}_s,G_s}$ for $0\leq s\leq 1$. Its important features are

1. p_s is a path of projections in $S(X,\varphi,Q)\otimes U(X,\varphi,P)$,
2. p_s arises from an ε'_X-partition $(\mathcal{F}_s,G_s)$, for all $0\leq s\leq 1$,
3. $p_0=p_{\mathcal{F},G}$ and
4. $p_1=\alpha_s\otimes\alpha_u(p_0)=p_{\mathcal{F}\circ\varphi^{-1},\varphi(G)}$.

Therefore $p_{\mathcal{F},G}$ and $p_{\mathcal{F}\circ\varphi^{-1},\varphi(G)}$ are homotopic projections in $S(X,\varphi,Q)\otimes U(X,\varphi,P)$.

Since $p_{\mathcal{F},G}$ and $p_{\mathcal{F}\circ\varphi^{-1},\varphi(G)}$ are homotopic, there is a partial isometry v in $S(X,\varphi,Q)\otimes U(X,\varphi,P)$ with initial projection $v^*v=p_{\mathcal{F},G}$ and final projection $vv^*=p_{\mathcal{F}\circ\varphi^{-1},\varphi(G)}$. By lemma 4.4.6 we have that $(u\otimes u)p_{\mathcal{F},G}(u^*\otimes u^*)=p_{\mathcal{F}\circ\varphi^{-1},\varphi(G)}$ and it is easy to check that the operator $\varrho=(u\otimes u)p_{\mathcal{F},G}v^*$ has the property $\varrho^*\varrho=\varrho\varrho^*=p_{\mathcal{F}\circ\varphi^{-1},\varphi(G)}$. Note that the operator ϱ is in $S\rtimes_{\alpha_s}\mathbb{Z}\otimes U\rtimes_{\alpha_u}\mathbb{Z}$ but not in $S(X,\varphi,Q)\otimes U(X,\varphi,P)$ since $u\otimes u$ is in the former but not the latter.

We are now ready to define a $*$-homomorphism $\delta:\mathscr{S}\to S\rtimes_{\alpha_s}\mathbb{Z}\otimes U\rtimes_{\alpha_u}\mathbb{Z}$. To do this, it suffices to define a partial isometry V in $S\rtimes_{\alpha_s}\mathbb{Z}\otimes U\rtimes_{\alpha_u}\mathbb{Z}$ with the same initial and final projection, $V^*V=VV^*$ is a projection. Then sending $z-1$ to $V-V^*V$ extends uniquely to such a map. (To see this, we simply note that $V+(1-V^*V)$ is a unitary in the unitization of the range. So there is a unique $*$-homomorphism mapping z in $C(S^1)$ to V, whose restriction to $\mathscr{S}\cong C^*(z-1)$ is as claimed). Since ϱ in $S\rtimes_{\alpha_s}\mathbb{Z}\otimes U\rtimes_{\alpha_u}\mathbb{Z}$

has the property that $\varrho^*\varrho = \varrho\varrho^* = p_{\mathcal{F}\circ\varphi^{-1},\varphi(G)}$, we obtain the required $*$-homomorphism, which we denote by δ.

4.4.7 Definition ([27])**.** The class δ in $KK(\mathscr{S}, S \rtimes_{\alpha_s} \mathbb{Z} \otimes U \rtimes_{\alpha_u} \mathbb{Z})$ is defined by the $*$ - homomorphism δ from $\mathscr{S}$ to $S \rtimes_{\alpha_s} \mathbb{Z} \otimes U \rtimes_{\alpha_u} \mathbb{Z}$ which is uniquely determined by $\delta(z-1) = \varrho - \varrho^*\varrho$.

4.4.2 The Second Duality Class

For the second duality class we construct an extension of $S \rtimes_{\alpha_s} \mathbb{Z} \otimes U \rtimes_{\alpha_u} \mathbb{Z}$.

Recall that $\mathcal{H} = \ell^2(X^h(P,Q))$. From section 3.2, we have representations $\pi_s : S(X,\varphi,Q) \to \mathcal{B}(\mathcal{H})$ and $\pi_u : U(X,\varphi,P) \to \mathcal{B}(\mathcal{H})$. From this point forwards we will suppress the representations. Moreover, the Ruelle algebras are also represented on $\mathcal{H}$, see section 3.3.

We begin this section by considering the manner in which operators in $S(X,\varphi,Q)$ and $U(X,\varphi,P)$ interact on $\mathcal{H}$.

4.4.8 Lemma ([34])**.** *If a is in $S(X,\varphi,Q)$ and b is in $U(X,\varphi,P)$, then ab and ba are compact operators on $\mathcal{H}$.*

Before we get to the proof, let us develop some geometric intuition using the hyperbolic toral automorphism, see section 2.3.2. Suppose $a_0 \in S(X,\varphi,Q)$ has support on a basic set $V^u(v,w,h^u,\delta)$ and $b_0 \in U(X,\varphi,P)$ has support on a basic set $V^s(v',w',h^s,\delta')$. So $Source(a_0) \subseteq X^u(w,\delta)$ and $Range(a_0) \subseteq X^u(v,\delta)$, and $Source(b_0) \subseteq X^s(w',\delta')$ and $Range(b_0) \subseteq X^s(v',\delta')$. See lemma 3.2.5 for further details. These sets are illustrated in figure 4.1 on page 53. Now we claim that a_0b_0 is a rank one operator. Indeed, we compute

$$a_0b_0\delta_x \quad = \quad a_0(h^u \circ h^s(x), h^s(x))b_0(h^s(x), x)\delta_{h^u\circ h^s(x)}$$

if $x \in X^s(w',\delta')$, $h^s(x) \in X^s(v',\delta')$, $h^s(x) \in X^u(w,\delta)$, and $h^u \circ h^s(x) \in X^u(v,\delta)$. The operator is zero otherwise. Therefore, from the picture below we see that x is the only possible point in X where a_0b_0 is non-zero. Whence, a_0b_0 is a rank one operator. Now every element $a \in S(X, \varphi, Q)$ and $b \in U(X, \varphi, P)$ is a norm limit of a finite sum of operators with supports on their respective basic sets. It follows that ab must be a compact operator.

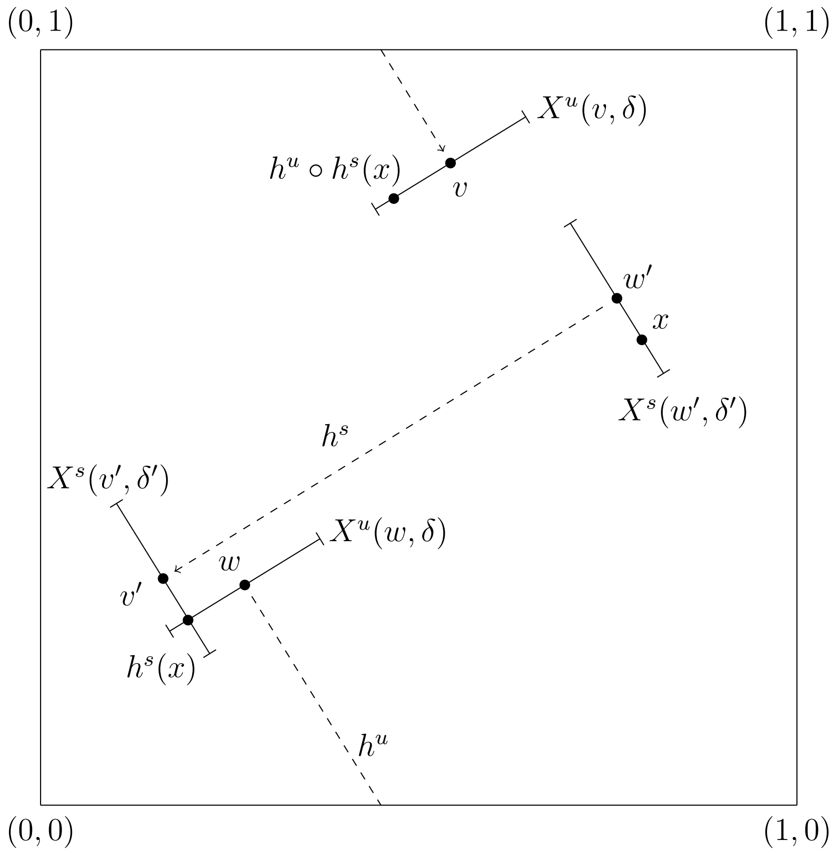


Figure 4.1: Hyperbolic Toral Automorphism

Proof. To begin, let us assume that both a in $S(X, \varphi, Q)$ and b in $U(X, \varphi, P)$ are supported on basic sets; that is, for $v, w \in X^u(Q)$ and $v', w' \in X^s(P)$, let the support of a be $V^u(v, w, h^u, \delta)$ and the support of b be $V^s(v', w', h^s, \delta')$. Note that $Source(a) \subseteq X^u(w, \delta)$ and $Range(a) \subseteq X^u(v, \delta)$, and $Source(b) \subseteq X^s(w', \delta')$ and $Range(b) \subseteq X^s(v', \delta')$. See lemma 3.2.5 for further details. We compute, for x in $X^h(P, Q)$,

$$a \cdot b\, \delta_x \;=\; a(h^u \circ h^s(x), h^s(x)) b(h^s(x), x) \delta_{h^u \circ h^s(x)}$$

if $x \in X^s(w', \delta')$, $h^s(x) \in X^s(v', \delta')$, $h^s(x) \in X^u(w, \delta)$, and $h^u \circ h^s(x) \in X^u(v, \delta)$.

Otherwise the product is zero. In particular, the product is zero unless $Range(b) \cap Source(a)$ is non-zero. However, uniqueness of the bracket implies that a local stable set and a local unstable set have non-trivial intersection at one point, at most. Whence, the product is zero unless $X^s(v', \delta')$ and $X^u(w, \delta)$ intersect and if they do the product is a rank one operator. Now finite sums of operators with supports as above form a dense set and therefore we obtain the compact operators by taking limits. Taking adjoints gives that $b \cdot a$ is also compact. □

4.4.9 Lemma. *If a is in $S(X, \varphi, Q)$ and b is in $U(X, \varphi, P)$, then*

$$\lim_{n\to+\infty} \alpha_s^{-n}(a) \cdot b = 0 \qquad \textit{and} \qquad \lim_{n\to+\infty} b \cdot \alpha_s^{-n}(a) = 0.$$

Before we begin the proof we note that the main idea is illustrated in figure 4.2 on page 55 for a hyperbolic toral automorphism with φ-invariant periodic points $P = \{(0,0)\}$ and $Q = \{(\frac{1}{2}, \frac{1}{2}), (0, \frac{1}{2}), (\frac{1}{2}, 0)\}$. Indeed, for $a \in S(X, \varphi, Q)$ supported on $X^u(Q)$ the figure demonstrates how the support of a is exponentially contracting while converging to points in Q. Now if b in $U(X, \varphi, P)$ is supported on a basic set then we can find neighbourhoods of the points in Q that do not intersect the support of b. Now for N sufficiently large the support of $\alpha^{-n}(a)$ and the support of b do not intersect for all $n \geq N$.

Proof. Let us assume that both a in $S(X, \varphi, Q)$ and b in $U(X, \varphi, P)$ are supported on basic sets; that is, for $v, w \in X^u(Q)$ and $v', w' \in X^s(P)$, let the support of a be $V^u(v, w, h^u, \delta)$ and the support of b be $V^s(v', w', h^s, \delta')$. Note that $Source(a) \subseteq X^u(w, \delta)$ and $Range(a) \subseteq X^u(v, \delta)$, and $Source(b) \subseteq X^s(w', \delta')$ and $Range(b) \subseteq X^s(v', \delta')$. See lemma 3.2.5 for further details.

We first aim to show that there exists N in $\mathbb{N}$ such that, for all $n \geq N$, we have $\alpha^{-n}(a) \cdot b = 0$. Indeed, from lemma 3.3.1 we have

$$\alpha_s^{-n}(a)\,\delta_z = a(h^u \circ \varphi^n(z), \varphi^n(z))\delta_{\varphi^{-n} \circ h^u \circ \varphi^n(z)}.$$

So we see that the support of $\alpha_s^{-n}(a)$ is $V^u(\varphi^{-n}(v), \varphi^{-n}(w), \varphi^{-n} \circ h^u \circ \varphi^{-n}, \lambda^{-n}\delta)$. Therefore, it follows that $Source(\alpha_s^{-n}(a)) \subseteq X^u(\varphi^{-n}(w), \lambda^{-n}\delta)$ and $Range(\alpha_s^{-n}(a)) \subseteq$

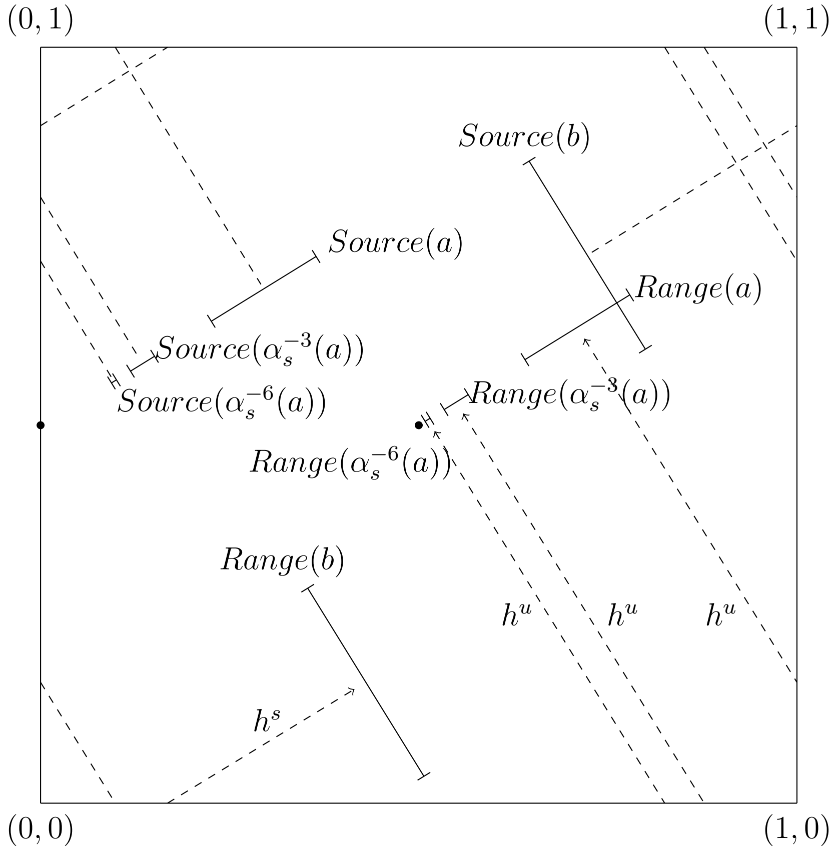


Figure 4.2: The support of $a \in S(X, \varphi, Q)$ is contracting under iterations of α^{-n} in a hyperbolic toral automorphism.

$X^u(\varphi^{-n}(v), \lambda^{-n}\delta)$. That is, the support of a is being exponentially contracted by repeated application of α_s. Moreover, we compute

$$\alpha_s^{-n}(a) \cdot b\,\delta_z \;=\; a(h^u \circ \varphi^n \circ h^s(z), \varphi^n \circ h^s(z))b(h^s(z), z)\delta_{\varphi^{-n} \circ h^u \circ \varphi^n \circ h^s(z)}$$

if $z \in X^s(w', \delta')$, $h^s(z) \in X^s(v', \delta')$, and $h^s(z) \in X^u(\varphi^{-n}(w), \lambda^{-n}\delta)$. It is zero otherwise. Now set $\varepsilon > 0$ small enough that $X^u(Q, \varepsilon) \cap X^s(v', \delta') = \varnothing$, we know this is possible since v' is in $X^s(P)$ while no point in Q is in $X^s(P)$ since P and Q are mutually distinct and φ-invariant. Given $\varepsilon > 0$, we can find an N in $\mathbb{N}$, such that $X^u(\varphi^{-n}(w), \lambda^{-n}\delta) \subset X^u(Q, \varepsilon)$ for all $n \geq \mathbb{N}$. This implies that, for all $n \geq \mathbb{N}$, we have $\alpha^{-n}(a) \cdot b = 0$. Now the general result follows since elements of $S(X, \varphi, Q)$ and $U(X, \varphi, P)$ are norm limits of linear combinations of elements with the above form. A similar argument gives the result for $b \cdot \alpha^{-n}(a)$. □

The following lemma appears in [28] and the main idea behind the proof appears in [34]. For completeness we reproduce the proof appearing in [28]. We also note that the main idea of the proof is illustrated, using a hyperbolic toral automorphism, in Figure 4.3 on page 59.

4.4.10 Lemma ([34] [28])**.** *For any a in $S(X,\varphi,Q)$ and b in $U(X,\varphi,P)$, we have*

$$\begin{aligned}
&\lim_{n\to\infty} \|\alpha_s^n(a)b - b\alpha_s^n(a)\| = 0,\\
&\lim_{n\to\infty} \|a\alpha_u^{-n}(b) - \alpha_u^{-n}(b)a\| = 0,\\
&\lim_{n\to\infty} \|\alpha_s^n(a)\alpha_u^{-n}(b) - \alpha_u^{-n}(b)\alpha_s^n(a)\| = 0.
\end{aligned}$$

Proof. We shall prove the third equality only; the others can be easily deduced. Indeed, to deduce the first equality from the third we compute

$$\begin{aligned}
0 &= \lim_{n\to\infty} \|\alpha_s^n(a)\alpha_u^{-n}(b) - \alpha_u^{-n}(b)\alpha_s^n(a)\|\\
&= \lim_{n\to\infty} \|u^n a u^{-n} u^{-n} b u^n - u^{-n} b u^n u^n a u^{-n}\|\\
&= \lim_{n\to\infty} \|u^{2n} a u^{-2n} b - b u^{2n} a u^{-2n}\|\\
&= \lim_{n\to\infty} \|u^{2n} a u^{-2n} b - b u^{2n} a u^{-2n}\|\\
&= \lim_{m\to\infty} \|\alpha_s^m(a)b - b\alpha_s^m(a)\|
\end{aligned}$$

where $m = 2n$. Similarly, we can show the second equality. We must now prove the third equality.

Let $\varepsilon > 0$ and assume that both a in $S(X,\varphi,Q)$ and b in $U(X,\varphi,P)$ are supported on basic sets; that is, for $v, w \in X^u(Q)$ and $v', w' \in X^s(P)$, let the support of a be $V^u(v,w,h^u,\delta)$ and the support of b be $V^s(v',w',h^s,\delta')$. Note that $Source(a) \subseteq X^u(w,\delta)$ and $Range(a) \subseteq X^u(v,\delta)$, and $Source(b) \subseteq X^s(w',\delta')$ and $Range(b) \subseteq X^s(v',\delta')$. See lemma 3.2.5 for further details. Using the computations similar to lemma 3.3.1 we have

$$\alpha_s^n(a) \cdot \alpha_u^{-n}(b)\,\delta_z =$$

$$a(h^u \circ \varphi^{-2n} \circ h^s \circ \varphi^n(z), \varphi^{-2n} \circ h^s \circ \varphi^n(z)) b(h^s \circ \varphi^n(z), \varphi^n(z)) \delta_{\varphi^n \circ h^u \circ \varphi^{-2n} \circ h^s \circ \varphi^n(z)}$$

$$\alpha_u^{-n}(b) \cdot \alpha_s^n(a)\, \delta_z =$$

$$b(h^s \circ \varphi^{2n} \circ h^u \circ \varphi^{-n}(z), \varphi^{2n} \circ h^u \circ \varphi^{-n}(z)) a(h^u \circ \varphi^{-n}(z), \varphi^{-n}(z)) \delta_{\varphi^{-n} \circ h^s \circ \varphi^{2n} \circ h^u \circ \varphi^{-n}(z)}.$$

Moreover, lemma 2.2 in [34] states that, there exists N such that for all $n \geq N$ we have,

$$\varphi^n \circ h^u \circ \varphi^{-2n} \circ h^s \circ \varphi^n(z) = \varphi^{-n} \circ h^s \circ \varphi^{2n} \circ h^u \circ \varphi^{-n}(z).$$

Now, suppose we are given z in $X^h(P, Q)$ such that $\varphi^{-n}(z) \in Source(a)$ and $\varphi^n(z) \in Source(b)$. We may define the following points:

$$\begin{aligned}
x_1 &= z \\
x_2 &= \varphi^n \circ h^u \circ \varphi^{-n}(z) \\
x_3 &= \varphi^n \circ h^u \circ \varphi^{-2n} \circ h^s \circ \varphi^n(z) = \varphi^{-n} \circ h^s \circ \varphi^{2n} \circ h^u \circ \varphi^{-n}(z) \\
x_4 &= \varphi^{-n} \circ h^s \circ \varphi^n(z).
\end{aligned}$$

In fact, given $\varepsilon_1 > 0$ and any z satisfying the above conditions, we can set N sufficiently large that we have, for all $n \geq N$:

$$\begin{aligned}
&x_2 \in X^s(x_1, \varepsilon_1) \qquad x_4 \in X^u(x_1, \varepsilon_1), \\
&x_1 \in X^s(x_2, \varepsilon_1) \qquad x_3 \in X^u(x_2, \varepsilon_1), \\
&x_4 \in X^s(x_3, \varepsilon_1) \qquad x_2 \in X^u(x_3, \varepsilon_1), \\
&x_3 \in X^s(x_4, \varepsilon_1) \qquad x_1 \in X^u(x_4, \varepsilon_1).
\end{aligned}$$

We have illustrated the relationship between these points for the hyperbolic toral automorphism in Figure 4.3 on page 59. Since a and b are taking basis vectors to basis vectors, we have

$$\|\alpha_s^n(a) \cdot \alpha_u^{-n}(b) - \alpha_u^{-n}(b) \cdot \alpha_s^n(a)\|$$

$$= \sup_z |a(\varphi^{-n}(x_3), \varphi^{-n}(x_4)) b(\varphi^n(x_4), \varphi^n(x_1)) - b(\varphi^n(x_3), \varphi^n(x_2)) a(\varphi^{-n}(x_2), \varphi^{-n}(x_1))|.$$

Now a and b are uniformly continuous so we may choose N large enough that the above condition is satisfied (the two versions of x_3 are equal) and so that ε_1 is sufficiently small that we have, for all $n \geq N$,

$$|a(\varphi^{-n}(x_3), \varphi^{-n}(x_4)) - a(\varphi^{-n}(x_2), \varphi^{-n}(x_1))| < \frac{\varepsilon}{2\|b\|}$$
$$|b(\varphi^{n}(x_4), \varphi^{n}(x_1)) - b(\varphi^{n}(x_3), \varphi^{n}(x_2))| < \frac{\varepsilon}{2\|a\|}.$$

Now we compute

$$\begin{aligned}
&|a(\varphi^{-n}(x_3), \varphi^{-n}(x_4))b(\varphi^{n}(x_4), \varphi^{n}(x_1)) - a(\varphi^{-n}(x_2), \varphi^{-n}(x_1))b(\varphi^{n}(x_3), \varphi^{n}(x_2))| \\
&= |a(\varphi^{-n}(x_3), \varphi^{-n}(x_4))b(\varphi^{n}(x_4), \varphi^{n}(x_1)) - a(\varphi^{-n}(x_3), \varphi^{-n}(x_4))b(\varphi^{n}(x_3), \varphi^{n}(x_2)) \\
&\quad + a(\varphi^{-n}(x_3), \varphi^{-n}(x_4))b(\varphi^{n}(x_3), \varphi^{n}(x_2)) - a(\varphi^{-n}(x_2), \varphi^{-n}(x_1))b(\varphi^{n}(x_3), \varphi^{n}(x_2))| \\
&\leq |a(\varphi^{-n}(x_3), \varphi^{-n}(x_4))||b(\varphi^{n}(x_4), \varphi^{n}(x_1)) - b(\varphi^{n}(x_3), \varphi^{n}(x_2))| \\
&\quad + |a(\varphi^{-n}(x_3), \varphi^{-n}(x_4)) - a(\varphi^{-n}(x_2), \varphi^{-n}(x_1))||b(\varphi^{n}(x_3), \varphi^{n}(x_2))| \\
&< \frac{\|a\|\varepsilon}{2\|a\|} + \frac{\|b\|\varepsilon}{2\|b\|} = \varepsilon.
\end{aligned}$$

Therefore, we have shown that

$$\lim_{n\to\infty} \|\alpha_s^n(a)\alpha_u^{-n}(b) - \alpha_u^{-n}(b)\alpha_s^n(a)\| = 0$$

which completes the proof. □

This completes the interactions of $S(X, \varphi, Q)$ and $U(X, \varphi, P)$ on $\mathcal{H}$. Our goal is to produce an extension of $S \rtimes_{\alpha_s} \mathbb{Z} \otimes U \rtimes_{\alpha_u} \mathbb{Z}$. To accomplish this we must represent each of these C^*-algebras as operators on a Hilbert space such that they commute modulo compact operators. Now let $f \in S \rtimes_{\alpha_s} \mathbb{Z}$ and $g \in U \rtimes_{\alpha_u} \mathbb{Z}$, one of the corollaries of the previous lemmas is that fg is equivalent to the zero operator modulo the compact operators on $\mathcal{H}$. Therefore, we must define a new Hilbert space. This is accomplished by taking an infinite direct sum of copies of $\mathcal{H}$. Define

$$\overline{\mathcal{H}} = \mathcal{H} \otimes \ell^2(\mathbb{Z}) = \bigoplus_{n\in\mathbb{Z}} \mathcal{H}.$$

We will define representations of $S \rtimes_{\alpha_s} \mathbb{Z}$ and $U \rtimes_{\alpha_u} \mathbb{Z}$ as bounded operators on $\overline{\mathcal{H}}$.

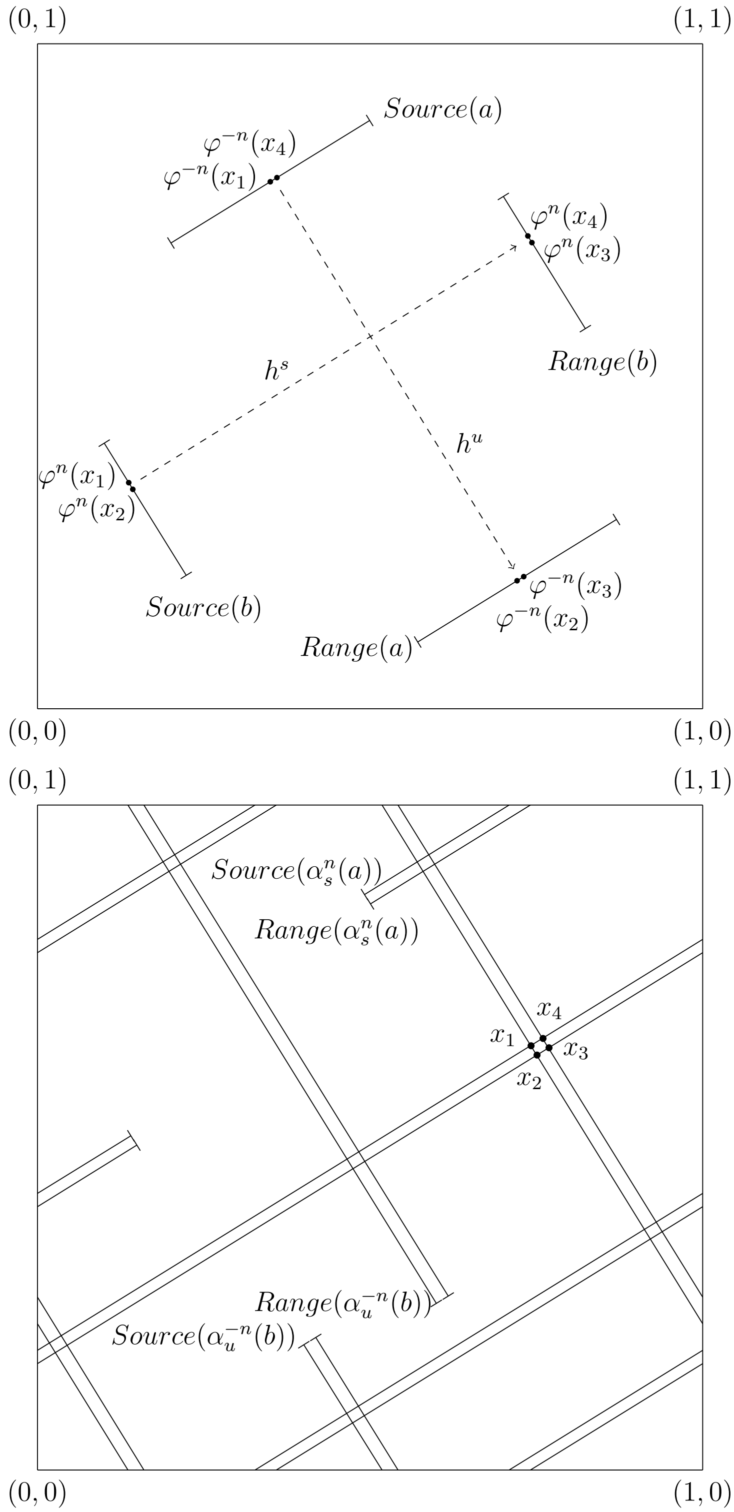


Figure 4.3: The points x_1, x_2, x_3, x_4 for a hyperbolic toral automorphism.

Recall that for δ_x in $\mathcal{H}$ we have the unitary operator $u\delta_x = \delta_{\varphi(x)}$ and $\alpha_s(a) = uau^*$ and $\alpha_u(b) = ubu^*$. The bilateral shift on $\ell^2(\mathbb{Z})$ will be denoted by B and is the operator given by $B\delta_n = \delta_{n-1}$. We note that from this point forwards we will always use δ_m and δ_n as basis vectors of $\ell^2(\mathbb{Z})$ and δ_x, δ_y and δ_z as basis vectors of $\mathcal{H} = \ell^2(X^h(P,Q))$.

Define $\overline{\pi}_s : S \rtimes_{\alpha_s} \mathbb{Z} \to \mathcal{B}(\mathcal{H} \otimes \ell^2(\mathbb{Z}))$, for a in $S(X,\varphi,Q)$, via

$$\overline{\pi}_s(a) = \bigoplus_{n\in\mathbb{Z}} \alpha_s^n(a) \qquad\qquad \overline{\pi}_s(u) = 1 \otimes B.$$

Also define $\overline{\pi}_u : U \rtimes_{\alpha_u} \mathbb{Z} \to \mathcal{B}(\mathcal{H} \otimes \ell^2(\mathbb{Z}))$, for b in $U(X,\varphi,P)$, via

$$\overline{\pi}_u(b) = b \otimes 1 \qquad\qquad \overline{\pi}_u(u) = u \otimes B^*.$$

4.4.11 Lemma. *The map $\overline{\pi}_s : S \rtimes_{\alpha_s} \mathbb{Z} \to \mathcal{B}(\overline{\mathcal{H}})$ is a representation.*

Proof. All the properties for the representation are obviously satisfied except for covariance. Let δ_x be a basis vector on $\mathcal{H}$ and δ_n a basis vector on $\ell^2(\mathbb{Z})$. Let a in $S(X,\varphi,Q)$. Using lemma 3.3.1, we compute

$$\overline{\pi}_s(\alpha_s(a)) = \overline{\pi}_s(uau^*) = \overline{\pi}_s(u)\overline{\pi}_s(a)\overline{\pi}_s(u)^*.$$

Thus, covariance is maintained. □

Similarly we have the analogous result for the unstable Ruelle algebra.

4.4.12 Lemma. *The map $\overline{\pi}_u : U \rtimes_{\alpha_u} \mathbb{Z} \to \mathcal{B}(\overline{\mathcal{H}})$ is a representation.*

So we have represented both Ruelle algebras as bounded operators on the Hilbert space $\overline{\mathcal{H}}$. We now take a close look at how all four C^*-algebras, $S(X,\varphi,Q)$, $U(X,\varphi,P)$, $S \rtimes_{\alpha_s} \mathbb{Z}$, and $U \rtimes_{\alpha_u} \mathbb{Z}$, interact on $\overline{\mathcal{H}}$. This leads to our main result of the section, an extension of $S \rtimes_{\alpha_s} \mathbb{Z} \otimes U \rtimes_{\alpha_u} \mathbb{Z}$.

4.4.13 Lemma. *If a is in $S(X,\varphi,Q)$ and b is in $U(X,\varphi,P)$, then $\overline{\pi}_s(a)\overline{\pi}_u(b)$ and $\overline{\pi}_u(b)\overline{\pi}_s(a)$ are never compact operators on $\overline{\mathcal{H}}$ unless either a or b is the zero operator.*

Proof. Let us tackle the mixing case first. If (X, d, φ) is a mixing Smale space then the representations $\overline{\pi}_s$ and $\overline{\pi}_u$ are faithful since the C^*-algebras $S(X, \varphi, Q)$ and $U(X, \varphi, P)$ are simple [37], see Section 3.2. Therefore, $\overline{\pi}_s(S(X, \varphi, Q)) \otimes \overline{\pi}_u(U(X, \varphi, P))$ is also simple. Thus, we must show that at least one non-zero operator is not compact.

In the irreducible case we recall that $S(X, \varphi, Q)$ and $U(X, \varphi, P)$ are finite direct sums of m simple algebras, $m \in \mathbb{N}$. Therefore, $S(X, \varphi, Q)) \otimes U(X, \varphi, P)$ is a direct sum of m^2 simple components. Now if $a \in S(X, \varphi, Q)$ has support in component i and $b \in U(X, \varphi, P)$ has support in component j we wish to consider these operators in the nth summand of the Hilbert space $\overline{\mathcal{H}}$. Notice that $\alpha^{m+i}(a)$ is back in the ith component since α is cyclically permuting a with period m. Therefore, the rank in summand n tends to infinity provided $n - i + j$ is a multiple of m, and the rank is zero in the other summands.

We now show that there is at least one non-compact operator. Select $p \in P$ and $q \in Q$ periodic points and let m be as above. Set $0 < \delta < \delta' < \varepsilon_X/2$ and define $b \in C_c(G^u(X, \varphi, P))$ to be supported on the basic set $V^s(p, p, id, \delta')$ such that $b(x, x) = 1$ for all $x \in X^s(p, \delta)$. Similarly, define $a \in C_c(G^s(X, \varphi, Q))$ to be supported on the basic set $V^u(q, q, id, \delta')$ such that $a(y, y) = 1$ for all $y \in X^u(q, \delta)$. Combining Lemma 5.9 and Proposition 5.12 in [28] it follows that there exists $N \in \mathbb{N}$ such that, for all $n \geq N$,

$$\varphi^{mn}(X^u(q, \delta)) \cap X^s(p, \delta) \neq \emptyset.$$

Now for any fixed $n \geq N$ with z in the above intersection, we compute

$$\|\alpha^{mn}(a) b \delta_z\| = \|\alpha^{mn}(a)\delta_z\| = \|a(\varphi^{-mn}(z), \varphi^{-mn}(z))\delta_z\| = \|\delta_z\| = 1.$$

Therefore, for every $n \geq N$ we have that $\|ab\|_{op} \geq 1$ on the mn^{th} summand of $\overline{\mathcal{H}} = \bigoplus_{n\in\mathbb{Z}} \mathcal{H}$. It follows that ab is not a compact operator on $\overline{\mathcal{H}}$.

Taking adjoints shows that $\overline{\pi}_u(b) \cdot \overline{\pi}_s(a)$ is also never compact. □

4.4.14 Lemma. *For any f in $S \rtimes_{\alpha_s} \mathbb{Z}$ and g in $U \rtimes_{\alpha_u} \mathbb{Z}$, we have*

$$[\overline{\pi}_s(f), \overline{\pi}_u(g)] = \overline{\pi}_s(f)\overline{\pi}_u(g) - \overline{\pi}_u(g)\overline{\pi}_s(f)$$

is a compact operator on $\overline{\mathcal{H}}$.

Proof. From lemma 4.4.8 we know that on each coordinate of $\overline{\mathcal{H}}$, for a in $S(X,\varphi,Q)$ and b in $U(X,\varphi,P)$, we have $\overline{\pi}_s(a)\overline{\pi}_u(b)$ and $\overline{\pi}_u(b)\overline{\pi}_s(a)$ are compact operators. However, by lemma 4.4.13, $\overline{\pi}_s(a)\overline{\pi}_u(b)$ and $\overline{\pi}_u(b)\overline{\pi}_s(a)$ is never a compact operator on $\overline{\mathcal{H}}$ unless either a or b is the zero operator. Denote the nth coordinate of

$$\overline{\mathcal{H}} = \bigoplus_{n\in\mathbb{Z}} \mathcal{H}$$

by $\overline{\mathcal{H}}_n$ and set $\varepsilon > 0$. Lemma 4.4.9 implies that there exists N_1 such that for $n \geq N_1$ we have that both $\|\alpha^{-n}(a)b\| < \varepsilon/2$ and $\|b\alpha^{-n}(a)\| < \varepsilon/2$. Therefore,

$$\begin{aligned}
\|(\overline{\pi}_s(a)\overline{\pi}_u(b) - \overline{\pi}_u(b)\overline{\pi}_s(a))|_{\overline{\mathcal{H}}_{-n}}\| &= \|\alpha^{-n}(a)b - b\alpha^{-n}(a)\| \\
&= \|\alpha^{-n}(a)b\| + \|b\alpha^{-n}(a)\| < \varepsilon.
\end{aligned}$$

Moreover, Lemma 4.4.10 implies that there exists N_2 such that for $n \geq N_2$ we have

$$\|\overline{\pi}_s(a)\overline{\pi}_u(b) - \overline{\pi}_u(b)\overline{\pi}_s(a))|_{\overline{\mathcal{H}}_n}\| = \|\alpha^n(a)b - b\alpha^n(a)\| < \varepsilon.$$

Therefore, for $a \in S(X,\varphi,Q)$ and $b \in U(X,\varphi,P)$ we have $[\overline{\pi}_s(a), \overline{\pi}_u(b)]$ is compact. Now the conclusion follows provided we show $\overline{\pi}_s(a)$ commutes with $\overline{\pi}_u(u)$, $\overline{\pi}_u(b)$ commutes with $\overline{\pi}_s(u)$, and $\overline{\pi}_u(u)$ commutes with $\overline{\pi}_s(u)$. Indeed, for $\delta_x \otimes \delta_n$ a basis vector on $\overline{\mathcal{H}}$, we compute

$$\begin{aligned}
\overline{\pi}_s(a)\overline{\pi}_u(u)\delta_x \otimes \delta_n &= \overline{\pi}_s(a)\delta_{\varphi(x)} \otimes \delta_{n+1} \\
&= \alpha^{n+1}(a)\delta_{\varphi(x)} \otimes \delta_{n+1} \\
&= u\alpha^n(a)u^*\delta_{\varphi(x)} \otimes \delta_{n+1} \\
&= u\alpha^n(a)\delta_x \otimes \delta_{n+1} \\
\overline{\pi}_u(u)\overline{\pi}_s(a)\delta_x \otimes \delta_n &= \overline{\pi}_u(u)(\alpha^n(a)\delta_x \otimes \delta_n) \\
&= u\alpha^n(a)\delta_x \otimes \delta_{n+1}).
\end{aligned}$$

Whence, $[\overline{\pi}_s(a), \overline{\pi}_u(u)] = 0$. Furthermore,

$$\begin{aligned}\overline{\pi}_u(b)\overline{\pi}_s(u) &= (b \otimes 1)(1 \otimes B) = (1 \otimes B)(b \otimes 1) = \overline{\pi}_s(u)\overline{\pi}_u(b)\\ \overline{\pi}_u(u)\overline{\pi}_s(u) &= (u \otimes B^*)(1 \otimes B) = u \otimes 1 = (1 \otimes B)(u \otimes B^*) = \overline{\pi}_s(u)\overline{\pi}_u(u).\end{aligned}$$

Thus, we also have $[\overline{\pi}_u(b), \overline{\pi}_s(u)] = 0$ and $[\overline{\pi}_u(u), \overline{\pi}_s(u)] = 0$. □

Define $\mathcal{E}$ to be the C^*-algebra generated by $\overline{\pi}_s(S \rtimes_{\alpha_s} \mathbb{Z})$, $\overline{\pi}_u(U \rtimes_{\alpha_u} \mathbb{Z})$, and $\mathcal{K}(\overline{\mathcal{H}})$. Note that neither $\overline{\pi}_s(S \rtimes_{\alpha_s} \mathbb{Z})$ or $\overline{\pi}_u(U \rtimes_{\alpha_u} \mathbb{Z})$ contain any compact operators on $\overline{\mathcal{H}}$ other than the zero operator. Now lemma 4.4.14 implies that $\overline{\pi}_s(S \rtimes_{\alpha_s} \mathbb{Z})$ and $\overline{\pi}_u(U \rtimes_{\alpha_u} \mathbb{Z})$ commute modulo the compact operators $\mathcal{K}(\overline{\mathcal{H}})$. From this we have that

$$\mathcal{E}/\mathcal{K}(\overline{\mathcal{H}}) \cong S \rtimes_{\alpha_s} \mathbb{Z} \otimes U \rtimes_{\alpha_u} \mathbb{Z}.$$

This gives the element Δ as follows.

4.4.15 Definition. The class Δ in $KK^1(S \rtimes_{\alpha_s} \mathbb{Z} \otimes U \rtimes_{\alpha_u} \mathbb{Z}, \mathbb{C})$ is given by the extension

$$0 \longrightarrow \mathcal{K}(\overline{\mathcal{H}}) \longrightarrow \mathcal{E} \longrightarrow S \rtimes_{\alpha_s} \mathbb{Z} \otimes U \rtimes_{\alpha_u} \mathbb{Z} \longrightarrow 0.$$

4.4.3 Proof of the Duality Theorem

In this section we will prove the Duality Theorem 4.4.1 for irreducible smale spaces. Let us begin with a summary of the proof to follow.

Summary of proof. Given the classes δ and Δ constructed in the previous two sections, we must show:

$$\delta \otimes_{U \rtimes_{\alpha_u} \mathbb{Z}} \Delta = \tau^{S \rtimes_{\alpha_s} \mathbb{Z}}(\mathcal{T}_0) \quad \text{and} \tag{4.10}$$

$$\delta \otimes_{S \rtimes_{\alpha_s} \mathbb{Z}} \Delta = -\tau^{U \rtimes_{\alpha_u} \mathbb{Z}}(\mathcal{T}_0). \tag{4.11}$$

In fact, we will only prove (4.10) since (4.11) is completely analogous.

We start with the class $\Delta \in KK^1(S \rtimes_{\alpha_s} \mathbb{Z} \otimes U \rtimes_{\alpha_u} \mathbb{Z}, \mathbb{C})$, from Definition 4.4.15, and the class $\delta \in KK^1(\mathbb{C}, S \rtimes_{\alpha_s} \mathbb{Z} \otimes U \rtimes_{\alpha_u} \mathbb{Z})$, appearing in Definition 4.4.7. We note that

Δ is given by an extension and δ is a $*$ - homomorphism from $\mathscr{S}$ to $S \rtimes_{\alpha_s} \mathbb{Z} \otimes U \rtimes_{\alpha_u} \mathbb{Z}$. Therefore, an application of lemma 4.1.3 gives us a class $\delta \otimes_{U \rtimes_{\alpha_u} \mathbb{Z}} \Delta$ in $KK^1(S \rtimes_{\alpha_s} \mathbb{Z} \otimes \mathscr{S}, S \rtimes_{\alpha_s} \mathbb{Z})$ given by the extension

$$0 \longrightarrow S \rtimes_{\alpha_s} \mathbb{Z} \otimes \mathcal{K}(\overline{\mathcal{H}}) \longrightarrow \mathcal{E}' \xrightarrow{\sigma_{12} \circ \pi'} S \rtimes_{\alpha_s} \mathbb{Z} \otimes \mathscr{S} \longrightarrow 0.$$

We are finished provided that this extension is a representative of the same KK-class as $\tau^{S \rtimes_{\alpha_s} \mathbb{Z}}(\mathcal{T}_0)$. The first step is to construct a class $\Theta \in KK(S \rtimes_{\alpha_s} \mathbb{Z} \otimes \mathscr{S}, S \rtimes_{\alpha_s} \mathbb{Z} \otimes \mathscr{S})$ which homotopic to $1_{S \rtimes_{\alpha_s} \mathbb{Z} \otimes \mathscr{S}}$ which we view as a type of untwisting. Now the product $\Theta \otimes_{S \rtimes_{\alpha_s} \mathbb{Z} \otimes \mathscr{S}} (\delta \otimes_{U \rtimes_{\alpha_u} \mathbb{Z}} \Delta)$ remains in the same KK-class as $\delta \otimes_{U \rtimes_{\alpha_u} \mathbb{Z}} \Delta$. Finally, we construct a unitary $\mathcal{U}$ and compute

$$ad_{\mathcal{U}}(\Theta \otimes_{S \rtimes_{\alpha_s} \mathbb{Z} \otimes \mathscr{S}} (\delta \otimes_{U \rtimes_{\alpha_u} \mathbb{Z}} \Delta))$$

which also does not change the KK-class. We then show that this is the class $\tau^{S \rtimes_{\alpha_s} \mathbb{Z}}(\mathcal{T}_0)$. □

We now begin with the formal proof of duality. We begin by introducing some (non-standard) notation that should help with understanding for some of the computations to follow.

Notation. We will represent the C^*-algebra $S \rtimes_{\alpha_s} \mathbb{Z} \otimes U \rtimes_{\alpha_u} \mathbb{Z} \otimes S \rtimes_{\alpha_s} \mathbb{Z}$ as bounded operators on the Hilbert space $\mathcal{H} \otimes \mathcal{H} \otimes \ell^2(\mathbb{Z})$. We will index the tensor products as $f \otimes_1 g \otimes_2 \xi$. Secondly, we have the operator

$$1 \otimes_1 \bigoplus_{n \in \mathbb{Z}} \alpha_s^n(a) \in \mathcal{B}(\mathcal{H} \otimes \mathcal{H} \otimes \ell^2(\mathbb{Z})),$$

and we want to simplify notation. Thus, we define

$$1 \otimes_1 \alpha_s^n(a) \otimes_2 e_n = 1 \otimes_1 \bigoplus_{n \in \mathbb{Z}} \alpha^n(a),$$

where e_n is thought of as a placeholder in nth coordinate of $\ell^2(\mathbb{Z})$.

4.4.16 Lemma. *The class of $\delta \otimes_{U \rtimes_{\alpha_u} \mathbb{Z}} \Delta$ in $KK^1(S \rtimes_{\alpha_s} \mathbb{Z} \otimes \mathscr{S}, S \rtimes_{\alpha_s} \mathbb{Z})$ is given by* *the extension*

$$0 \longrightarrow S \rtimes_{\alpha_s} \mathbb{Z} \otimes \mathcal{K}(\overline{\mathcal{H}}) \longrightarrow \mathcal{E}' \xrightarrow{\sigma_{12} \circ \pi'} S \rtimes_{\alpha_s} \mathbb{Z} \otimes \mathscr{S} \longrightarrow 0.$$

Proof. We start with the class $\Delta \in KK^1(S \rtimes_{\alpha_s} \mathbb{Z} \otimes U \rtimes_{\alpha_u} \mathbb{Z}, \mathbb{C})$, from Definition 4.4.15, given by the extension

$$0 \longrightarrow \mathcal{K}(\overline{\mathcal{H}}) \longrightarrow \mathcal{E} \xrightarrow{\pi} S \rtimes_{\alpha_s} \mathbb{Z} \otimes U \rtimes_{\alpha_u} \mathbb{Z} \longrightarrow 0$$

and the class $\delta \in KK(\mathscr{S}, S \rtimes_{\alpha_s} \mathbb{Z} \otimes U \rtimes_{\alpha_u} \mathbb{Z})$, appearing in Definition 4.4.7, defined by a $*$ - homomorphism $\delta : \mathscr{S} \to S \rtimes_{\alpha_s} \mathbb{Z} \otimes U \rtimes_{\alpha_u} \mathbb{Z}$. Expanding the definition of the product we have

$$\delta \otimes_{U \rtimes_{\alpha_u} \mathbb{Z}} \Delta = \sigma_{12}(\tau_{\mathscr{S}} \tau_{S \rtimes_{\alpha_s} \mathbb{Z}}(\delta) \otimes \tau^{S \rtimes_{\alpha_s} \mathbb{Z}} \sigma_{12}(\Delta)).$$

We will step through the process of this product.

The extension $\tau^{S \rtimes_{\alpha_s} \mathbb{Z}} \sigma_{12}(\Delta) \in KK^1(S \rtimes_{\alpha_s} \mathbb{Z} \otimes U \rtimes_{\alpha_u} \mathbb{Z} \otimes S \rtimes_{\alpha_s} \mathbb{Z}, S \rtimes_{\alpha_s} \mathbb{Z})$ is given by

$$0 \to S \rtimes_{\alpha_s} \mathbb{Z} \otimes \mathcal{K}(\overline{\mathcal{H}}) \to S \rtimes_{\alpha_s} \mathbb{Z} \otimes \mathcal{E} \xrightarrow{1 \otimes \pi} S \rtimes_{\alpha_s} \mathbb{Z} \otimes U \rtimes_{\alpha_u} \mathbb{Z} \otimes S \rtimes_{\alpha_s} \mathbb{Z} \to 0.$$

Note that we have a representation of $S \rtimes_{\alpha_s} \mathbb{Z} \otimes U \rtimes_{\alpha_u} \mathbb{Z} \otimes S \rtimes_{\alpha_s} \mathbb{Z}$ to $\mathcal{B}(\mathcal{H} \otimes \mathcal{H} \otimes \ell^2(\mathbb{Z}))$ given by $1 \otimes_1 \overline{\pi}_u \otimes_2 \overline{\pi}_s$, defined in Section 4.4.2. These maps are given on generators by

$$\begin{array}{ll} a \otimes_1 1 \otimes_2 1 \mapsto a \otimes_1 1 \otimes_2 1 & u \otimes_1 1 \otimes_2 1 \mapsto u \otimes_1 1 \otimes_2 1 \\ 1 \otimes_1 b \otimes_2 1 \mapsto 1 \otimes_1 b \otimes_2 1 & 1 \otimes_1 u \otimes_2 1 \mapsto 1 \otimes_1 u \otimes_2 B^* \\ 1 \otimes_1 1 \otimes_2 a \mapsto 1 \otimes_1 \alpha^n(a) \otimes_2 e_n & 1 \otimes_1 1 \otimes_2 u \mapsto 1 \otimes_1 1 \otimes_2 B, \end{array}$$

where we have suppressed the representations. See section 4.4.2 for details on these operators.

The class $\tau_{\mathscr{S}} \tau_{S \rtimes_{\alpha_s} \mathbb{Z}}(\delta)$ is given by a $*$-homomorphism

$$\delta \otimes 1 : \mathscr{S} \otimes S \rtimes_{\alpha_s} \mathbb{Z} \longrightarrow S \rtimes_{\alpha_s} \mathbb{Z} \otimes U \rtimes_{\alpha_u} \mathbb{Z} \otimes S \rtimes_{\alpha_s} \mathbb{Z}.$$

For convenience, we note that $\delta\otimes 1$ is uniquely determined by the map sending $z-1\otimes a\cdot u^k$ into $S\rtimes_{\alpha_s}\mathbb{Z}\otimes U\rtimes_{\alpha_u}\mathbb{Z}\otimes S\rtimes_{\alpha_s}\mathbb{Z}$ via the maps

$$
\begin{aligned}
z\otimes 1 &\mapsto (u\otimes_1 u\otimes_2 1)(p_{\mathcal{F},G}\otimes_2 1)(v^*\otimes_2 1),\\
1\otimes a &\mapsto (u\otimes_1 u\otimes_1 1)(p_{\mathcal{F},G}\otimes_2 1)(u^*\otimes_1 u^*\otimes_2 1)(1\otimes_1 1\otimes_2 a),\\
1\otimes u &\mapsto (u\otimes_1 u\otimes_2 1)(p_{\mathcal{F},G}\otimes_2 1)(u^*\otimes_1 u^*\otimes_2 1)(1\otimes_1 1\otimes_2 u).
\end{aligned}
$$

By lemma 4.1.3, the Kasparov product $\tau_{\mathscr{S}}\tau_{S\rtimes_{\alpha_s}\mathbb{Z}}(\delta)\otimes\tau^{S\rtimes_{\alpha_s}\mathbb{Z}}\sigma_{12}(\Delta)$ is given by the extension

$$0\longrightarrow S\rtimes_{\alpha_s}\mathbb{Z}\otimes\mathcal{K}(\overline{\mathcal{H}})\longrightarrow\mathcal{E}'\xrightarrow{\pi'}\mathscr{S}\otimes S\rtimes_{\alpha_s}\mathbb{Z}\longrightarrow 0$$

where $\mathcal{E}'$ is the C^*-algebra generated by

$$(\delta\otimes 1)\circ(1\otimes\overline{\pi}_u\otimes\overline{\pi}_s):\mathscr{S}\otimes S\rtimes_{\alpha_s}\mathbb{Z}\to\mathcal{B}(\mathcal{H}\otimes\mathcal{H}\otimes\ell^2(\mathbb{Z}))\quad\text{and}\quad S\rtimes_{\alpha_s}\mathbb{Z}\otimes\mathcal{K}(\overline{\mathcal{H}}).$$

Furthermore, the extension $\sigma_{12}(\tau_{\mathscr{S}}\tau_{S\rtimes_{\alpha_s}\mathbb{Z}}(\delta)\otimes\tau^{S\rtimes_{\alpha_s}\mathbb{Z}}\sigma_{12}(\Delta))$ is given by

$$0\longrightarrow S\rtimes_{\alpha_s}\mathbb{Z}\otimes\mathcal{K}(\overline{\mathcal{H}})\longrightarrow\mathcal{E}'\xrightarrow{\sigma_{12}\circ\pi'}S\rtimes_{\alpha_s}\mathbb{Z}\otimes\mathscr{S}\longrightarrow 0.$$

□

Now that we have computed the product $\delta\otimes_{U\rtimes_{\alpha_u}\mathbb{Z}}\Delta$, we must show that it defines the same class in $KK(S\rtimes_{\alpha_s}\mathbb{Z}\otimes\mathscr{S}\otimes\mathscr{S},S\rtimes_{\alpha_s}\mathbb{Z})$ as $\tau^{S\rtimes_{\alpha_s}\mathbb{Z}}(\mathcal{T}_0)$. The first part of this process is an untwisting step. In particular, we shall define an automorphism $\Theta:S\rtimes_{\alpha_s}\mathbb{Z}\otimes\mathscr{S}\to S\rtimes_{\alpha_s}\mathbb{Z}\otimes\mathscr{S}$ which is homotopic to $1_{S\rtimes_{\alpha_s}\mathbb{Z}\otimes\mathscr{S}}$ in $KK(S\rtimes_{\alpha_s}\mathbb{Z}\otimes\mathscr{S},S\rtimes_{\alpha_s}\mathbb{Z}\otimes\mathscr{S})$. Indeed, for $a\in S(X,\varphi,Q)$, u implementing the action α_s, and $f(t)\in\mathscr{S}=C_0(0,1)$, define

$$\Theta(a\cdot u^k\otimes f(t))=a\cdot u^k\otimes e^{2\pi ikt}f(t).$$

We note that this map is given on generators by

$$1\otimes z\mapsto 1\otimes z\qquad,\qquad a\otimes 1\mapsto a\otimes 1\qquad,\qquad u\otimes 1\mapsto u\otimes z.$$

At this point we need to accomplish two things. First, we must show that Θ is homotopic to $1_{S\rtimes_{\alpha_s}\mathbb{Z}}$ and second, we must compute the product

$$\Theta \otimes_{(S\rtimes_{\alpha_s}\mathbb{Z})\otimes\mathscr{S}} (\delta \otimes_{U\rtimes_{\alpha_u}\mathbb{Z}} \Delta).$$

4.4.17 Lemma. *There exists an automorphism $\Theta : S\rtimes_{\alpha_s}\mathbb{Z}\otimes\mathscr{S} \to S\rtimes_{\alpha_s}\mathbb{Z}\otimes\mathscr{S}$ defined on generators by*

$$\Theta(a \cdot u^k \otimes f(t)) = a \cdot u^k \otimes e^{2\pi ikt} f(t).$$

Moreover, Θ is the identity in $KK(S \rtimes_{\alpha_s} \mathbb{Z} \otimes \mathscr{S}, S \rtimes_{\alpha_s} \mathbb{Z} \otimes \mathscr{S})$

Proof. We will show a homotopy from Θ to the identity. Indeed, for $r \in [0,1]$ define,

$$\Theta_r(a \cdot u^k \otimes f(t)) = a \cdot u^k \otimes e^{2\pi ikrt} f(t).$$

We want to show that Θ_r is a $*$-automorphism for every r in $[0,1]$ and t in $(0,1)$. To see that covariance is maintained, for all r in $[0,1]$ and t in $(0,1)$, we compute

$$\begin{aligned} \Theta_r(u\otimes 1)\Theta_r(a\otimes f(t))\Theta_r(u\otimes 1)^* &= (u\otimes e^{2\pi irt})(a\otimes f(t))(u^*\otimes e^{-2\pi irt}) \\ &= (uau^*\otimes f(t) = \alpha_s(a)\otimes f(t) = \Theta_r(\alpha_s(a)\otimes f(t)). \end{aligned}$$

Therefore, the map Θ_r satisfies the covariance conditions for all $r \in [0,1]$ and t in $(0,1)$, so extends to a $*$-homomorphism on $S \rtimes_{\alpha_s} \mathbb{Z} \otimes \mathscr{S}$. We can explicitly write a formula for the inverse of Θ_r on generators so that Θ_r is actually a $*$-automorphism. Moreover, Θ_r is clearly is faithful since $S \rtimes_{\alpha_s} \mathbb{Z}\otimes\mathscr{S}$ is simple and Θ is clearly onto. Now we must show that each map

$$a \cdot u^k \otimes f(t) \mapsto \Theta_r(a \cdot u^k \otimes f(t)) = a \cdot u^k \otimes e^{2\pi iktr} f(t)$$

is continuous. Let $\varepsilon > 0$, and set $\delta = \frac{\varepsilon}{kM}$ where M is the maximum value of $\|a \cdot u^k \otimes f(t)\|$

for $t \in (0,1)$. For $|r-r'|<\delta$ we compute

$$\begin{aligned}
\|\Theta_r(a\cdot u^k\otimes f(t)) - \Theta_{r'}(a\cdot u^k\otimes f(t))\| &= \|a\cdot u^k\otimes e^{2\pi iktr}f(t) - a\cdot u^k\otimes e^{2\pi iktr'}f(t)\| \\
&= \|a\cdot u^k\otimes(1-e^{2\pi ikt(r'-r)})f(t)\| \\
&= |1-e^{2\pi ikt(r'-r)}|\|a\cdot u^k\otimes f(t)\| \\
&< \frac{\varepsilon}{M}\|a\cdot u^k\otimes f(t)\| \le \varepsilon.
\end{aligned}$$

Finally,

$$\begin{aligned}
\Theta_0(a\cdot u^k\otimes f(t)) &= a\cdot u^k\otimes e^{2\pi ik0t}f(t) = a\cdot u^k\otimes f(t) \\
\Theta_1(a\cdot u^k\otimes f(t)) &= a\cdot u^k\otimes e^{2\pi ik1t}f(t) = \Theta(a\cdot u^k\otimes f(t))
\end{aligned}$$

so that $\Theta_0 = \mathrm{Id}$ and $\Theta_1 = \Theta$. Therefore, Θ is homotopic to $1_{S\rtimes_{\alpha_s}\mathbb{Z}\otimes\mathscr{S}}$ in $KK(S\rtimes_{\alpha_s}\mathbb{Z}\otimes\mathscr{S}, S\rtimes_{\alpha_s}\mathbb{Z}\otimes\mathscr{S})$. □

4.4.18 Lemma. *The class of* $\Theta\otimes_{S\rtimes_{\alpha_s}\mathbb{Z}\otimes\mathscr{S}}(\delta\otimes_{U\rtimes_{\alpha_u}\mathbb{Z}}\Delta)$ *in* $KK^1(S\rtimes_{\alpha_s}\mathbb{Z}\otimes\mathscr{S}, S\rtimes_{\alpha_s}\mathbb{Z})$ *is given by the extension*

$$0 \longrightarrow S\rtimes_{\alpha_s}\mathbb{Z}\otimes\mathcal{K}(\overline{\mathcal{H}}) \longrightarrow \mathcal{E}'' \xrightarrow{\sigma_{12}\circ\pi''} S\rtimes_{\alpha_s}\mathbb{Z}\otimes\mathscr{S} \longrightarrow 0.$$

Furthermore, this extension represents the same class in KK*-theory as* $\delta\otimes_{U\rtimes_{\alpha_u}\mathbb{Z}}\Delta$.

Proof. To begin we note that, by lemma 4.4.16,

$$\delta\otimes_{U\rtimes_{\alpha_u}\mathbb{Z}}\Delta = \sigma_{12}(\tau_{\mathscr{S}}\tau_{S\rtimes_{\alpha_s}\mathbb{Z}}(\delta)\otimes\tau^{S\rtimes_{\alpha_s}\mathbb{Z}}\sigma_{12}(\Delta)) \in KK^1(S\rtimes_{\alpha_s}\mathbb{Z}\otimes\mathscr{S}, S\rtimes_{\alpha_s}\mathbb{Z})$$

is given by the extension

$$0 \longrightarrow S\rtimes_{\alpha_s}\mathbb{Z}\otimes\mathcal{K}(\overline{\mathcal{H}}) \longrightarrow \mathcal{E}' \xrightarrow{\sigma_{12}\circ\pi'} S\rtimes_{\alpha_s}\mathbb{Z}\otimes\mathscr{S} \longrightarrow 0.$$

Now since Θ is an automorphism, an application of lemma 4.1.3 implies that $\Theta\otimes_{S\rtimes_{\alpha_s}\mathbb{Z}\otimes\mathscr{S}}(\delta\otimes_{U\rtimes_{\alpha_u}\mathbb{Z}}\Delta)$ is given by the extension

$$0 \longrightarrow S\rtimes_{\alpha_s}\mathbb{Z}\otimes\mathcal{K}(\overline{\mathcal{H}}) \longrightarrow \mathcal{E}'' \xrightarrow{\sigma_{12}\circ\pi''} S\rtimes_{\alpha_s}\mathbb{Z}\otimes\mathscr{S} \longrightarrow 0$$

where $\mathcal{E}'' = \{x \in \mathcal{E}' \mid \sigma_{12} \circ \pi''(x) = \Theta^{-1}(\pi'(x))\}$. Moreover, the final statement in the lemma follows from lemma 4.4.17. □

At this point, let us take a moment to summarize our result so far. Combining lemma 4.4.16 and lemma 4.4.18 we have the following composition of maps:

$$S \rtimes_{\alpha_s} \mathbb{Z} \otimes \mathscr{S} \xrightarrow{\quad\Theta\quad} S \rtimes_{\alpha_s} \mathbb{Z} \otimes \mathscr{S} \xrightarrow{\quad\sigma_{12}\quad} \mathscr{S} \otimes S \rtimes_{\alpha_s} \mathbb{Z}$$
$$\xrightarrow{\quad\delta \otimes 1\quad} S \rtimes_{\alpha_s} \mathbb{Z} \otimes U \rtimes_{\alpha_u} \mathbb{Z} \otimes S \rtimes_{\alpha_s} \mathbb{Z}$$
$$\xrightarrow{\quad 1 \otimes \overline{\pi}_u \otimes \overline{\pi}_s\quad} \mathcal{B}(\mathcal{H} \otimes \mathcal{H} \otimes \ell^2(\mathbb{Z})).$$

Let us denote this composition by $\psi : S \rtimes_{\alpha_s} \mathbb{Z} \otimes \mathscr{S} \to \mathcal{B}(\mathcal{H} \otimes \mathcal{H} \otimes \ell^2(\mathbb{Z}))$ and observe that $\psi(S \rtimes_{\alpha_s} \mathbb{Z} \otimes \mathscr{S}) \subset \mathcal{E}''$. Moreover, since $S \rtimes_{\alpha_s} \mathbb{Z} \otimes \mathscr{S}$ is generated by $a \cdot u^k \otimes z^l - 1$ where $a \in S(X, \varphi, Q)$, we note this map is given on generators by

$$\begin{aligned}
1 \otimes z &\mapsto ((u \otimes_1 u)p_{\mathcal{F},G}v^*) \otimes_2 B^*,\\
a \otimes 1 &\mapsto ((u \otimes_1 u)p_{\mathcal{F},G}(u \otimes_1 u)^*) \otimes_2 1)(1 \otimes_1 \alpha^n(a) \otimes_2 e_n),\\
u \otimes 1 &\mapsto ((u \otimes_1 u)p_{\mathcal{F},G}v^*) \otimes_2 1.
\end{aligned}$$

To complete the proof we must show that the class

$$\Theta \otimes_{S \rtimes_{\alpha_s} \mathbb{Z} \otimes \mathscr{S}} (\delta \otimes_{U \rtimes_{\alpha_u} \mathbb{Z}} \Delta)$$

in $KK^1(S \rtimes_{\alpha_s} \mathbb{Z} \otimes \mathscr{S}, S \rtimes_{\alpha_s} \mathbb{Z})$ is equivalent to $\tau^{S \rtimes_{\alpha_s} \mathbb{Z}}(\mathcal{T}_0)$. The equivalence we produce uses a slightly modified version of the unitary produced in the work of Kaminker and Putnam [27].

Suppose that $(\mathcal{F}, G)$ is an ϵ'_X-partition of X, as in section 4.4.7. We define a vector, which we denote χ_G in $\mathcal{H} = \ell^2(X^h(P, Q))$ which takes the value $(\#G)^{-1/2}$ on the set G and zero elsewhere. Note that χ_G is a unit vector. We let q_G denote the rank one projection onto the span of χ_G.

We define $W_{\mathcal{F},G}$ by setting, for all y in $X^h(P,Q)$,

$$W_{\mathcal{F},G}\delta_y \otimes \chi_G = \sum_k f_k(y)\delta_{[y,g_k]} \otimes \delta_{[g_k,y]},$$

where the sum is taken over all k such that y is in $B(g_k, \varepsilon'_X/2)$. Recall that in an ε'_X-partition of X the support of f_k is contained in $B(g_k, \varepsilon'_X/2)$. Using our standard convention (the bracket returns zero if it is not defined), we will simply write the sum above as being over all $k = 1, 2, \ldots, K$ since $f_k(y)$ will be zero if $[y, g_k]$ and $[g_k, y]$ fail to be defined. We also set $W_{\mathcal{F},G}\delta_z \otimes \xi = 0$, for ξ in $\mathcal{H}$ orthogonal to χ_G.

It is easy to verify that

$$W^*_{\mathcal{F},G}\delta_y \otimes \delta_z = \begin{cases} f_k([y,z])\delta_{[y,z]} \otimes \chi_G & \text{if } y \in X^u(g_k, \varepsilon), z \in X^s(g_k, \varepsilon) \\ 0 & \text{otherwise} \end{cases}$$

for all w, z in $X^h(P,Q)$. The following lemma summarizes the basic properties of W.

4.4.19 Lemma ([27]). *Suppose that $(\mathcal{F}, G)$ is an ϵ'_X-partition of X and $W_{\mathcal{F},G}$ is defined as above. Then*

1. $W^*_{\mathcal{F},G}W_{\mathcal{F},G} = 1 \otimes q_G$.
2. $W_{\mathcal{F},G}W^*_{\mathcal{F},G} = p_{\mathcal{F},G}$.
3. *If $(\mathcal{F} \circ \varphi^{-1}, \varphi(G))$ is also an ε'_X-partition, then*

$$(u \otimes u)W_{\mathcal{F},G}(u \otimes u)^* = W_{\mathcal{F}\circ\varphi^{-1},\varphi(G)}.$$

4. $W_{\mathcal{F},G}(S(X,\varphi,Q) \otimes \mathcal{K}) \subset S(X,\varphi,Q) \otimes \mathcal{K}$.

Proof. Let us drop the subscripts and set $W = W_{\mathcal{F},G}$. For the first item we compute

$$\begin{aligned} W^*W\delta_y \otimes \chi_G &= \begin{cases} \sum_k f_k(y)W^*(\delta_{[y,g_k]} \otimes \delta_{[g_k,y]}) & \text{if } y \in B(g_k, \varepsilon'_X/2) \\ 0 & \text{otherwise} \end{cases} \\ &= \sum_k f_k(y) f_k([[y,g_k],[g_k,y]])\delta_{[[y,g_k],[g_k,y]]} \otimes \chi_G \\ &= \sum_k f_k^2(y)\delta_y \otimes \chi_G \\ &= \delta_y \otimes \chi_G. \end{aligned}$$

For the second item we have a similar computation:

$$WW^*\delta_y \otimes \delta_z = \begin{cases} f_i([y,z])W(\delta_{[y,z]} \otimes \chi_G) & \text{if } y \in X^u(g_i,\varepsilon), z \in X^s(g_i,\varepsilon) \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} f_i([y,z]) \sum_k f_k([y,z])\delta_{[[y,z],g_k]} \otimes \delta_{[g_k,[y,z]]} & \text{if } y \in X^u(g_i,\varepsilon), z \in X^s(g_i,\varepsilon) \\ 0 & \text{otherwise} \end{cases}$$

$$= f_i([y,z]) \sum_k f_k([y,z])\delta_{[y,g_k]} \otimes \delta_{[g_k,z]}$$

$$= p_{\mathcal{F},G}(\delta_y \otimes \delta_z).$$

The third item follows directly from the definitions. Finally, for the fourth item, let a be in $S(X,\varphi,Q)$ and suppose k is any compact operator. Now

$$W(a \otimes k) = W(1 \otimes q_G)(a \otimes k) = W(a \otimes q_G)(1 \otimes k),$$

and so it suffices to show $W(a \otimes q_G)$ is in $S(X,\varphi,Q) \otimes \mathcal{K}$. The C^*-algebra $S(X,\varphi,Q)$ has an approximate identity consisting of continuous functions of compact support on $X^u(Q)$. Moreover, such functions are spanned by elements supported on sets of the form $X^u(v,\epsilon'_X/2)$. So it suffices to consider a point v in $X^u(Q)$, a function a in $S(X,\varphi,Q)$ supported on a basic set of the form $V^u(v,v,h^u,\epsilon'_X/2)$ such that $a\delta_y = a(y,y)\delta_y$ if y is in $X^u(v,\epsilon'_X/2)$ and zero otherwise, and prove that $W(a \otimes q_G)$ is in $S(X,\varphi,Q) \otimes \mathcal{K}$. For each k, let us define a function b_k supported on a basic set of the form $V^u([v,g_k],v,h^u,\epsilon'_X/2)$ by $b_k(y',y) = a(y,y)f_k(y)$ if $d(y,g_k) < \epsilon'_X$ and $[y,g_k] = y'$ and to be zero otherwise. Also define e_k to be the rank one operator which maps χ_G to $\delta_{[g_k,v]}$ and is zero on the orthogonal complement of χ_G. It follows that b_k is in $S(X,\varphi,Q)$. Now we compute

$$\begin{aligned} W(a\otimes q_G)(\delta_y\otimes\chi_G) &= a(y,y)W(\delta_y\otimes\chi_G)\\ &= a(y,y)\sum_k f_k(y)\delta_{[y,g_k]}\otimes\delta_{[g_k,y]}\\ &= \sum_k a(y,y)f_k(y)\delta_{[y,g_k]}\otimes\delta_{[g_k,y]}\\ &= \sum_k b_k\otimes e_k(\delta_y\otimes\chi_G). \end{aligned}$$

Therefore, $W(a\otimes q_G)$ is in $S(X,\varphi,Q)\otimes\mathcal{K}(\mathcal{H})$. □

4.4.20 Lemma ([27])**.** *Let a be in $S(X,\varphi,Q)$ and let $(\mathcal{F},G)$ be an ε'_X-partition. Then we have*

$$\lim_{n\to\infty}\|(1\otimes\alpha_s^n(a))W_{\mathcal{F},G}-W_{\mathcal{F},G}(\alpha_s^n(a)\otimes 1)\|=0.$$

Proof. Again, let us drop the subscripts and set $W=W_{\mathcal{F},G}$. It suffices to prove the result for a supported in a basic set of the form $V^u(v,w,h^u,\delta)$ and further, since we are taking limits as n goes to positive infinity, we may also assume that v and w are within $\varepsilon'_X/2$ so that h^u is given by the bracket map.

We observe that both operators $(1\otimes\alpha_s^n(a))W$ and $W(1\otimes\alpha_s^n(a))$ are zero on the orthogonal complement of $\ell^2(X^h(P,Q))\otimes\mathbb{C}\cdot\chi_G$. We consider y in $\mathcal{H}=\ell^2(X^h(P,Q))$ and compute

$$\begin{aligned} (1\otimes\alpha_s^n(a))W(\delta_y\otimes\chi_G) &= (1\otimes\alpha_s^n(a))\sum_k f_k(y)\delta_{[y,g_k]}\otimes\delta_{[g_k,y]}\\ &= \sum_k f_k(y)a([\varphi^{-n}[g_k,y],v],\varphi^{-n}[g_k,y])\delta_{[y,g_k]}\otimes\delta_{\varphi^n[\varphi^{-n}[g_k,y],v]} \end{aligned}$$

and also

$$\begin{aligned} W(\alpha_s^n(a)\otimes 1)(\delta_y\otimes\chi_G) &= Wa([\varphi^{-n}(y),v],\varphi^{-n}(y))\delta_{\varphi^n[\varphi^{-n}(y),v]}\otimes\chi_P\\ &= \sum_k f_k(\varphi^n[\varphi^{-n}(y),v])a([\varphi^{-n}(y),v],\varphi^{-n}(y))\delta_{[\varphi^n[\varphi^{-n}(y),v],g_k]}\otimes\delta_{[g_k,\varphi^n[\varphi^{-n}(y),v]]}. \end{aligned}$$

Let $\varepsilon>0$ be given. Let M be an upper bound on the function $|a|$. We may find a constant $\varepsilon_1 > 0$ such that $|f_k(y) - f_k(z)| < \epsilon/2MK$, for all y, z with $d(y, z) < \epsilon_1$ and $1 \le k \le K$. In addition, we select $\epsilon_1 > 0$ such that $|a([y, v], y) - a([z, v], z)| < \epsilon/2K$ for all y, z with z in $X^u(y, \epsilon_1)$. We choose N sufficiently large so that $\lambda^{-n}\epsilon_X/2 < \epsilon_1$ and $\lambda^{-n}\epsilon_X < \epsilon'_X$, for all $n \ge N$. With $n \ge N$, holding k fixed for the moment, we make the claim that if the coefficient in either expression above:

$$f_k(y)a([\varphi^{-n}[g_k, y], v], \varphi^{-n}[g_k, y]) \qquad \text{or} \qquad f_k(\varphi^n[\varphi^{-n}(y), v])a([\varphi^{-n}(y), v], \varphi^{-n}(y))$$

is not zero, then we have

1. $[y, g_k] = [\varphi^n[\varphi^{-n}(y), v], g_k]$,
2. $\varphi^n[\varphi^{-n}[g_k, y], v] = [g_k, \varphi^n[\varphi^{-n}(y), v]]$,
3. the map sending (y, g_k) to $([y, g_k], \varphi^n[\varphi^{-n}[g_k, y], v])$ is injective,
4. $\varphi^{-n}[g_k, y]$ is in $X^u(\varphi^{-n}(y), \epsilon_1)$,
5. $d(\varphi^n[\varphi^{-n}(y), v], y) < \epsilon_1$.

If $f_k(y)a([\varphi^{-n}[g_k, y], v], \varphi^{-n}[g_k, y])$ is non-zero, then $f_k(y)$ must be non-zero and this means that y is in $B(g_k, \epsilon'_X/2)$. Moreover, from the choice of ϵ'_X, we have $[g_k, y]$ is in $X^u(y, \epsilon_X/2)$ and hence $\varphi^{-n}([g_k, y])$ is in $X^u(\varphi^{-n}(y)\lambda^{-n}\epsilon_X/2)$. In addition, we know that $a([\varphi^{-n}[g_k, y], v], \varphi^{-n}[g_k, y])$ is non-zero and this means that $\varphi^{-n}([g_k, y])$ is in $X^u(w, \epsilon'_X/2)$ and it follows that $\varphi^{-n}(y)$ is in $X^u(w, \epsilon'_X/2+\lambda^{-n}\epsilon_X/2)$. Since $\lambda^{-n}\epsilon_X < \epsilon'_X < \epsilon_X/2$, we see that $[\varphi^{-n}(y), v]$ is also defined and is in $X^s(\varphi^{-n}(y), \epsilon_X/2)$. It follows that $\varphi^n[\varphi^{-n}(y), v]$ is in $X^s(y, \lambda^{-n}\epsilon_X/2)$ and also in $B(g_k, \epsilon'_X)$.

If the second expression, $f_k(\varphi^n[\varphi^{-n}(y), v])a([\varphi^{-n}(y), v], \varphi^{-n}(y))$ is non-zero, then we must have that $a([\varphi^{-n}(y), v], \varphi^{-n}(y))$ is non-zero and so $\varphi^{-n}(y)$ is in $X^u(w, \epsilon'_X/2)$. Then we have $[\varphi^{-n}(y), v]$ is in $X^s(\varphi^{-n}(y), \epsilon_X/2)$ and hence $\varphi^n[\varphi^{-n}(y), v]$ is in $X^s(y, \lambda^{-n}\epsilon_X/2)$. In addition, if the coefficient is non-zero, the f_k term is non-zero and this means that this same point is in $B(g_k, \epsilon'_X/2)$ and hence, y is in $B(g_k, \epsilon'_X/2+\lambda^{-n}\epsilon_X/2)$. Since $\lambda^{-n}\epsilon_X < \epsilon'_X$, $[g_k, y]$ is in $X^u(y, \epsilon_X/2)$.

To summarize, if either term is non-zero, then we have $[g_k, y]$ is defined and is in

$X^u(y, \epsilon_X/2)$, $\varphi^{-n}(y)$ is in $X^u(w, \epsilon'_X)$ and $[\varphi^{-n}(y), v]$ is defined and in $X^s(\varphi^{-n}(y), \epsilon_X/2)$. Parts 4 and 5 of the claim follow at once since $\lambda^{-n}\epsilon_X/2 < \epsilon_1$.

For any $0 \leq m \leq n$, we have

$$d(\varphi^{m-n}(y), \varphi^m[\varphi^{-n}(y), x]) \leq \lambda^{-m} d(\varphi^{-n}(y), [\varphi^{-n}(y), v]) \leq \lambda^{-m}\epsilon_X/2 \leq \epsilon_X/2,$$

and

$$d(\varphi^{m-n}(y), \varphi^{m-n}[g_k, y]) \leq \lambda^{-n+m} d(y, [g_k, y]) \leq \lambda^{-n+m}\epsilon_X/2 \leq \epsilon_X/2.$$

From the triangle inequality, we have

$$d(\varphi^m[\varphi^{-n}(y), v], \varphi^{m-n}[g_k, y]) \leq \epsilon_X.$$

This means that the bracket of these points is defined (in either order). First, taking bracket in the order given and using the φ-invariance of the bracket we have

$$\begin{aligned} [\varphi^m[\varphi^{-n}(y), v], \varphi^{m-n}[g_k, y]] &= \varphi^m[[\varphi^{-n}(y), v], \varphi^{-n}[g_k, y]] \\ &= \varphi^m[\varphi^{-n}(y), \varphi^{-n}[g_k, y]]. \end{aligned}$$

When $m = n$, the left hand side becomes

$$[\varphi^n[\varphi^{-n}(y), v], \varphi^{n-n}[g_k, y]] = [\varphi^n[\varphi^{-n}(y), v], [g_k, y]] = [\varphi^n[\varphi^{-n}(y), v], y]]$$

while the right hand side is

$$\varphi^n[\varphi^{-n}(y), \varphi^{-n}[g_k, y]] = \varphi^n(\varphi^{-n}(y)) = y$$

as $\varphi^{-n}[g_k, y]$ is in $X^u(\varphi^{-n}(y), \lambda^{-n}\epsilon_X/2)$. Now bracketing each with g_k yields

$$[y, g_k] = [[\varphi^n[\varphi^{-n}(y), v], y], g_k] = [\varphi^n[\varphi^{-n}(y), v], g_k]$$

and we have established part 1 of the claim.

On the other hand, if we bracket in the other order, and again use the φ-invariance, we obtain

$$[\varphi^{m-n}[g_k, y], \varphi^m[\varphi^{-n}(y), v]] = \varphi^m[\varphi^{-n}[g_k, y], [\varphi^{-n}(y), v]] = \varphi^m[\varphi^{-n}[g_k, y], v].$$

Setting $m = n$, the left hand side is

$$[\varphi^{n-n}[g_k, y], \varphi^n[\varphi^{-n}(y), v]] = [[g_k, y], \varphi^n[\varphi^{-n}(y), v]] = [g_k, \varphi^n[\varphi^{-n}(y), v]$$

while the right is

$$\varphi^n[\varphi^{-n}[g_k, y], \varphi^{-n}(y)].$$

We have established the second part of the claim.

For the third part of the claim, let $x = [y, g_k]$ and $z = [g_k, \varphi^n[\varphi^{-n}(y), v]]$. We can recover y and g_k from x and z by observing that $[z, x] = g_k$ and

$$[x, z] = [[y, g_k], [g_k, \varphi^n[\varphi^{-n}(y), v]]] = [y, \varphi^n[\varphi^{-n}(y), v]] = y,$$

since $\varphi^n[\varphi^{-n}(y), v]$ is in $X^s(y, \epsilon)$.

We may conclude from the first two parts of our claim that

$$\begin{aligned}&(1 \otimes \alpha_s^n(a))W - W(\alpha_s^n(a) \otimes 1)(\delta_y \otimes \chi_G) \\ &\quad = \sum_k (f_k(y)a([\varphi^{-n}[g_k, y], v], \varphi^{-n}[g_k, y]) - f_k(\varphi^n[\varphi^{-n}(y), v])a([\varphi^{-n}(y), v], \varphi^{-n}(y))) \\ &\qquad\qquad \delta_{[y,g_k]} \otimes \delta_{[g_k, \varphi^n[\varphi^{-n}(y), v]]}.\end{aligned}$$

From part 3, we see that the vectors appearing in the right hand side of the expression at the end of the last paragraph are pairwise orthogonal and the sums obtained for different values of y are pairwise orthogonal. From this it follows that

$$\begin{aligned}&\|(1 \otimes \alpha_s^n(a))W - W(\alpha_s^n(a) \otimes 1)\|^2 \\ &\quad = \sup_{y \in X^h(P,Q)} \sum_k |f_k(y)a([\varphi^{-n}[g_k, y], v], \varphi^{-n}[g_k, y]) \\ &\qquad - f_k(\varphi^n[\varphi^{-n}(y), v])a([\varphi^{-n}(y), v], \varphi^{-n}(y))|^2.\end{aligned}$$

The estimate that, for fixed k and y,

$$|f_k(y)a([\varphi^{-n}[g_k,y],v],\varphi^{-n}[g_k,y]) - f_k(\varphi^n[\varphi^{-n}(y),v])a([\varphi^{-n}(y),v],\varphi^{-n}(y))| < \epsilon/K$$

follows from the last two parts of our claim and standard techniques. This completes the proof. $\square$

We next define

$$\tilde{W}_{\mathcal{F},G} = \begin{bmatrix} W_{\mathcal{F},G} & (1 - W_{\mathcal{F},G}W^*_{\mathcal{F},G})^{1/2} \\ -(1 - W^*_{\mathcal{F},G}W_{\mathcal{F},G})^{1/2} & W^*_{\mathcal{F},G} \end{bmatrix}$$

which is a unitary operator. Moreover, it follows from part 4 of Lemma 4.4.19 that it is in the multiplier algebra of $S(X,\varphi,Q) \otimes \mathcal{K}$.

4.4.21 Lemma ([27])**.** *Let a be in $S(X,\varphi,Q)$ and let $(\mathcal{F},G)$ be an ε-partition. Then we have*

$$\lim_{n\to+\infty} \|((p_{\mathcal{F},G}(1\otimes\alpha^n(a)))\otimes e_{1,1})\tilde{W}_{\mathcal{F},G} - \tilde{W}_{\mathcal{F},G}((\alpha^n(a)\otimes q_G)\otimes e_{1,1})\| = 0.$$

Moreover, if $(\mathcal{F}\circ\varphi^{-1},\varphi(G))$ is also an ε'_X-partition, then $\tilde{W}_{\mathcal{F}\circ\varphi^{-1},\varphi(G)}$ is also in the multiplier algebra of $S(X,\varphi,Q)\otimes\mathcal{K}$ and we have

$$\lim_{n\to+\infty} \|\tilde{W}^*_{\varphi(G)}((p_{\varphi(G)}(1\otimes\alpha^n(a)))\otimes e_{1,1}) - ((\alpha^n(a)\otimes q_{\varphi(G)})\otimes e_{1,1})\tilde{W}^*_{\varphi(G)}\| = 0$$

where $\tilde{W}_{\varphi(G)} = \tilde{W}_{\mathcal{F}\circ\varphi^{-1},\varphi(G)}$ and $p_{\varphi(G)} = p_{\mathcal{F}\circ\varphi^{-1},\varphi(G)}$.

Proof. Again, let us drop the subscripts and set $W = W_{\mathcal{F},G}$, $\tilde{W} = \tilde{W}_{\mathcal{F},G}$, and $p = p_{\mathcal{F},G}$. First, notice that p is in $S\otimes U$, while $\alpha^n(a)$ is in S. It follows from Lemma 4.4.10 that in the first term we have

$$\lim_{n\to\infty} \|p(1\otimes\alpha_s^n(a)) - (1\otimes\alpha_s^n(a))p\| = 0.$$

So it suffices to prove the result after interchanging the order of p and $1\otimes\alpha_s^n(a)$ in the first term.

It follows from the fact that $WW^* = p$ that the (new) first term above is

$$\begin{aligned}(((1 \otimes \alpha_s^n(a))p) \otimes e_{1,1})\tilde{W} &= ((1 \otimes \alpha_s^n(a)) \otimes e_{1,1})(p \otimes e_{1,1})\tilde{W} \\ &= ((1 \otimes \alpha_s^n(a)) \otimes e_{1,1})(W \otimes e_{1,1}) \\ &= (1 \otimes \alpha_s^n(a))W \otimes e_{1,1}.\end{aligned}$$

On the other hand, using the fact that $W^*W = 1 \otimes q_G$, the second term above is

$$\begin{aligned}\tilde{W}((\alpha_s^n(a) \otimes q_G) \otimes e_{1,1}) &= \tilde{W}((1 \otimes q_G) \otimes e_{1,1})(\alpha_s^n(a) \otimes 1) \otimes e_{1,1}) \\ &= (W \otimes e_{1,1})((\alpha_s^n(a) \otimes 1) \otimes e_{1,1}) \\ &= W((\alpha_s^n(a) \otimes 1) \otimes e_{1,1}).\end{aligned}$$

The first statement now follows at once from Lemma 4.4.20 and the second statement follows from combining the first statement with part 3 of lemma 4.4.19. □

At this point we are ready to prove the Duality Theorem 4.4.1. Suppose that $(\mathcal{F}, G)$ is an ϵ'_X-partition and $(\mathcal{F} \circ \varphi^{-1}, \varphi(G))$ is also an ε'_X-partition of X. For the remainder we will use the conventions $p_{\mathcal{F},G} = p_G$, $p_{\mathcal{F}\circ\varphi^{-1},\varphi(G)} = p_{\varphi(G)}$, $W_{\mathcal{F},G} = W_G$, and $W_{\mathcal{F}\circ\varphi^{-1},\varphi(G)} = W_{\varphi(G)}$.

The extension

$$0 \longrightarrow S \rtimes_{\alpha_s} \mathbb{Z} \otimes \mathcal{K}(\mathcal{H} \otimes \ell^2(\mathbb{Z})) \longrightarrow \mathcal{E}'' \longrightarrow S \rtimes_{\alpha_s} \mathbb{Z} \otimes \mathscr{S} \longrightarrow 0$$

is a representative of the class $\Theta \otimes_{\mathscr{S} \otimes S \rtimes_{\alpha_s} \mathbb{Z}} (\delta \otimes_{U \rtimes_{\alpha_u} \mathbb{Z}} \Delta)$ in $KK^1(S \rtimes_{\alpha_s} \mathbb{Z} \otimes \mathscr{S}, S \rtimes_{\alpha_s} \mathbb{Z})$ where $\mathcal{E}''$ is the C^*-algebra generated by $S \rtimes_{\alpha_s} \mathbb{Z} \otimes \mathcal{K}(\mathcal{H} \otimes \ell^2(\mathbb{Z}))$ and the closure of the image of the representation ψ sending $a \cdot u^k \otimes z^l - 1$ into $\mathcal{B}(\mathcal{H} \otimes \mathcal{H} \otimes \ell^2(\mathbb{Z}))$ defined by

$$\begin{aligned}\psi(a \cdot u^k \otimes z^l - 1) &= (((u \otimes_1 u)p_G v^*) \otimes_2 B^*)^{l+k}(1 \otimes_1 \alpha_s^n(a) \otimes_2 e_n)(p_{\varphi(G)} \otimes_2 B)^k \\ &\quad -(((u \otimes_1 u)p_G v^*) \otimes_2 B^*)^k(1 \otimes_1 \alpha_s^n(a) \otimes_2 e_n)(p_{\varphi(G)} \otimes_2 B)^k\end{aligned}$$

where a is in $S(X, \varphi, Q)$ and k, l are in $\mathbb{Z}$. Now we will conjugate the class $\Theta \otimes_{\mathscr{S} \otimes S \rtimes_{\alpha_s} \mathbb{Z}} (\delta \otimes_{U \rtimes_{\alpha_u} \mathbb{Z}} \Delta)$ by the unitary $\tilde{W}_{\varphi(G)} \otimes_2 1$ and show that the extension obtained is equivalent to $\tau^{S \rtimes_{\alpha_s} \mathbb{Z}}(\mathcal{T}_0)$. This will complete the proof.

Since the unitary $\tilde{W}_{\varphi(G)} \otimes_2 1$ is in $M_2(\mathcal{H} \otimes \mathcal{H} \otimes \ell^2(\mathbb{Z}))$ we define

$$\tilde{\psi} = \begin{pmatrix} \psi & 0 \\ 0 & 0 \end{pmatrix} : S \rtimes_{\alpha_s} \mathbb{Z} \otimes \mathscr{S} \to \mathcal{B}[(\mathcal{H} \otimes \mathcal{H} \otimes \ell^2(\mathbb{Z})) \bigoplus (\mathcal{H} \otimes \mathcal{H} \otimes \ell^2(\mathbb{Z}))].$$

and using this representation gives the same class in KK-theory. By the proof of Lemma 4.4.21 we have $(\tilde{W}^*_{\varphi(G)} \otimes_2 1)(\tilde{\psi}(a \cdot u^k \otimes z^l - 1))(\tilde{W}_{\varphi(G)} \otimes_2 1)$ is given by

$$[(W^*_{\varphi(G)} \otimes_2 1)(\psi(a \cdot u^k \otimes z^l - 1))(W_{\varphi(G)} \otimes_2 1)] \otimes e_{1,1}.$$

We compute, for a in $S(X, \varphi, Q)$ and k, l in $\mathbb{Z}$,

$$(W^*_{\varphi(G)} \otimes_2 1)(\psi(a \cdot u^k \otimes z^l - 1))(W_{\varphi(G)} \otimes_2 1)$$

$$= (W^*_{\varphi(G)} \otimes_2 1)(((u \otimes_1 u)p_G v^*) \otimes_2 B^*)^{l+k}(1 \otimes_1 \alpha_s^n(a) \otimes_2 e_n)(p_{\varphi(G)} \otimes_2 B)^k(W_{\varphi(G)} \otimes_2 1)$$
$$-(W^*_{\varphi(G)} \otimes_2 1)(((u \otimes_1 u)p_G v^*) \otimes_2 B^*)^{k}(1 \otimes_1 \alpha_s^n(a) \otimes_2 e_n)(p_{\varphi(G)} \otimes_2 B)^k(W_{\varphi(G)} \otimes_2 1)$$
$$= ((W^*_{\varphi(G)}(u\otimes_1 u)v^* W_{\varphi(G)})\otimes_2 B^*)^{l+k}(W^*_{\varphi(G)}\otimes_2 1)(1\otimes_1 \alpha_s^n(a)\otimes_2 e_n)(W_{\varphi(G)}\otimes_2 1)(1\otimes_1 1\otimes_2 B)^k$$
$$-((W^*_{\varphi(G)}(u\otimes_1 u)v^* W_{\varphi(G)})\otimes_2 B^*)^{k}(W^*_{\varphi(G)}\otimes_2 1)(1\otimes_1 \alpha^n(a)\otimes_2 e_n)(W_{\varphi(G)}\otimes_2 1)(1\otimes_1 1\otimes_2 B)^k.$$

Now observe that by combining lemmas 4.4.8 and 4.4.9 and using the operator $p_{\mathcal{F},G}$ in $S \otimes U$, we have, up to $S \rtimes_{\alpha_s} \mathbb{Z} \otimes \mathcal{K}(\overline{\mathcal{H}})$,

$$(W^*_{\varphi(G)} \otimes_2 1)(1 \otimes_1 \alpha_s^n(a) \otimes_2 e_n)(W_{\varphi(G)} \otimes_s 1))$$
$$= \begin{cases} (W^*_{\varphi(G)} \otimes_2 1)(1 \otimes_1 \alpha_s^n(a) \otimes_2 e_n)(W_{\varphi(G)} \otimes_2 1)) & \text{for } n \geq 0 \\ 0 & \text{for } n < 0. \end{cases}$$

Therefore, we can consider our Hilbert space to be $\mathcal{H} \otimes \mathcal{H} \otimes \ell^2(\mathbb{N})$. Moreover, up to $S \rtimes_{\alpha_s} \mathbb{Z} \otimes \mathcal{K}(\overline{\mathcal{H}})$ we have, by lemma 4.4.21, that

$$(W^*_{\varphi(G)} \otimes_2 1)(1 \otimes_1 \alpha_s^n(a) \otimes_2 e_n)(W_{\varphi(G)} \otimes_2 1)) = \alpha_s^n(a) \otimes_1 q_{\varphi(G)} \otimes_2 e_n.$$

Therefore, on the Hilbert space $\mathcal{B}(\mathcal{H} \otimes \mathcal{H} \otimes \ell^2(\mathbb{N}))$, we have the following operator:

$$\begin{aligned}
&(W^*_{\varphi(G)} \otimes_2 1)(\psi(a \cdot u^k \otimes z^l - 1))(W_{\varphi(G)} \otimes_2 1)\\
&= ((W^*_{\varphi(G)}(u \otimes_1 u)v^* W_{\varphi(G)}) \otimes_2 B^*)^{l+k}(\alpha_s^n(a) \otimes_1 q_{\varphi(G)} \otimes_2 e_n)(1 \otimes_1 1 \otimes_2 B)^k\\
&\quad -((W^*_{\varphi(G)}(u \otimes_1 u)v^* W_{\varphi(G)}) \otimes_2 B^*)^{k}(\alpha_s^n(a) \otimes_1 q_{\varphi(G)} \otimes_2 e_n)(1 \otimes_1 1 \otimes_2 B)^k.
\end{aligned}$$

We now observe that the quotient map $\pi'' : \mathcal{E}'' \to S \rtimes_{\alpha_s} \mathbb{Z} \otimes \mathscr{S}$ is defined by

$$\begin{aligned}
(W^*_{\varphi(G)}(u \otimes_1 u)v^* W_{\varphi(G)}) \otimes_2 1 &= u \otimes 1\\
\alpha^n(a) \otimes_1 q_{\varphi(G)} \otimes_2 e_n &= a \otimes 1\\
1 \otimes_1 1 \otimes_2 B &= u^* \otimes \overline{z}.
\end{aligned}$$

Let us compute

$$\begin{aligned}
&\pi''[(W^*_{\varphi(G)} \otimes_2 1)(\psi(a \cdot u^k \otimes z^l - 1))(W_{\varphi(G)} \otimes_2 1)]\\
&\qquad = (uu^* \otimes z)^{k+l}(a \otimes 1)(u \otimes \overline{z})^k - (uu^* \otimes z)^k(a \otimes 1)(u \otimes \overline{z})^k\\
&\qquad = (1 \otimes z^{k+l})(a \otimes 1)(u^k \otimes \overline{z}^k) - (1 \otimes z^k)(a \otimes 1)(u^k \otimes \overline{z}^k)\\
&\qquad = a \cdot u^k \otimes z^l - a \cdot u^k \otimes 1\\
&\qquad = a \cdot u^k \otimes z^l - 1.
\end{aligned}$$

Therefore, we see that the extension produced is a representative of the class $\tau^{S \rtimes_{\alpha_s} \mathbb{Z}}(\mathcal{T}_0)$. So we have shown that the class $\Theta \otimes_{\mathscr{S} \otimes S \rtimes_{\alpha_s} \mathbb{Z}} (\delta \otimes_{U \rtimes_{\alpha_u} \mathbb{Z}} \Delta)$ and $\tau^{S \rtimes_{\alpha_s} \mathbb{Z}}(\mathcal{T}_0)$ are equal in $KK^1(S \rtimes_{\alpha_s} \mathbb{Z} \otimes \mathscr{S}, S \rtimes_{\alpha_s} \mathbb{Z})$ and the Duality Theorem 4.4.1 is proven.

Chapter 5

Spectral Triples for Smale Spaces

5.1 Spectral Triples

Here we define a spectral triple and state some general properties of spectral triples used in the sequel. For a general reference to spectral triples see Connes' excellent book [8].

To simplify notation we begin to employ $[a, b]$ to denote the commutator $ab - ba$.

5.1.1 Definition. A *spectral triple* $(A, \mathcal{H}, D)$ consists of

(i) a separable Hilbert space $\mathcal{H}$,

(ii) a $*$-algebra A of bounded operators on $\mathcal{H}$,

(iii) an unbounded self-adjoint operator D on $\mathcal{H}$ such that:

- **(a)** the set $\{a \in A | [D, a] \in \mathcal{B}(\mathcal{H})\}$ is norm dense in A and
- **(b)** the operator $a(1 + D^2)^{-1}$ is a compact operator on $\mathcal{H}$ for all a in A.

A spectral triple $(A, \mathcal{H}, D)$ is said to be *even* if there is a grading operator $\Gamma \in \mathcal{H}$ such that

$$\Gamma = \Gamma^* \quad , \quad \Gamma^2 = 1 \quad , \quad D\Gamma + \Gamma D = 0 \quad , \quad a\Gamma = \Gamma a$$

for all $a \in A$. Otherwise it is *odd*.

Let us make some remarks about the definition. Whenever one works with unbounded operators there are subtleties, and the above definition is no exception. We should technically say D is an unbounded operator from $Domain(D)$ to $\mathcal{H}$, then we also require in part (a) that $a : Domain(D) \to Domain(D)$ and $[D, a]$ extends from a bounded operator on $Domain(D)$ to a bounded operator on $\mathcal{H}$. We also recall that addition and subtraction of unbounded operators occurs on the intersection of their respective domains and that a self-adjoint operator has dense domain (by definition). Secondly, we note that the set $\{a \in A | [D, a] \in \mathcal{B}(\mathcal{H})\}$ is always a $*$-algebra. Indeed, if $[D, a]$ is a bounded operator, then so is $[D, a]^* = -[D, a^*]$ so that a^* is also an element of the set. Moreover, if both $[D, a]$ and $[D, b]$ are bounded then

$$[D, ab] = Dab - abD = Dab - aDb + aDb - abD = [D, a]b - a[D, b] \in \mathcal{B}(\mathcal{H}).$$

Finally, we also note that the condition that $a(1 + D^2)^{-1}$ is a compact operator on $\mathcal{H}$ for all a in A can be replaced with $(1 + D^2)^{-1}$ is a compact operator on $\mathcal{H}$, when A is unital.

We will work exclusively with ungraded spectral triples in this dissertation. An ungraded spectral triple naturally defines a class in the K-homology group $KK^1(A, \mathbb{C})$. We remind the reader that a class in K-homology is given by a Fredholm module. For a fantastic account of K-homology we refer the reader to [24]. Let us recall the definition of a Fredholm module and then state the theorem giving a Fredholm module from a spectral triple.

5.1.2 Definition. An (ungraded) Fredholm module over a C^*-algebra A is a triple $(A, \mathcal{H}, F)$ such that

(i) $\mathcal{H}$ is a separable Hilbert space,

(ii) A is represented as bounded operators on $\mathcal{H}$, and

(iii) F is a bounded operator on $\mathcal{H}$ such that, for every $a \in A$,

$$(F^2 - 1)a \in \mathcal{K}(\mathcal{H}), \quad (F - F^*)a \in \mathcal{K}(\mathcal{H}), \quad Fa - aF \in \mathcal{K}(\mathcal{H}).$$

A Fredholm module $(A, \mathcal{H}, F)$ is said to be *even* if there is a grading operator $\Gamma \in \mathcal{H}$ such that

$$\Gamma = \Gamma^* \quad , \quad \Gamma^2 = 1 \quad , \quad F\Gamma + \Gamma F = 0 \quad , \quad a\Gamma = \Gamma a$$

for all $a \in A$. Otherwise it is *odd*. Moreover, we say a Fredholm module is *contractive* if $\|F\| \leq 1$ and *self-adjoint* if $F = F^*$.

5.1.3 Theorem ([2, 9, 25])**.** *Let A be a C^*-algebra. If $(A, \mathcal{H}, D)$ is an ungraded spectral triple, then $(A, \mathcal{H}, F)$ is a Fredholm module and defines an element of $KK^1(A, \mathbb{C})$ where $F = D(1 + D^2)^{-\frac{1}{2}}$. Furthermore, $(A, \mathcal{H}, F)$ is self-adjoint and contractive.*

Spectral triples carry more information than Fredholm modules exactly in the fact that we can look at the growth rate of the spectrum of the operator D. In the unital case a simple method of doing this is to consider whether the compact operator $(1+D^2)^{-\frac{p}{2}}$ is trace class for some $p > 0$. A weaker notion is to consider whether the operator $e^{-t(1+D^2)}$ is trace class for all $t > 0$. We define these notions of summability.

5.1.4 Definition. Suppose $(A, \mathcal{H}, D)$ is a spectral triple over a unital C^*-algebra A with

$$\mathrm{Tr}((1 + D^2)^{-\frac{p}{2}}) < \infty$$

for some positive number p. Then the spectral triple is said to be *p-summable*. Furthermore, the value

$$\dim_S((A, \mathcal{H}, D)) := \inf\{p > 0 | \mathrm{Tr}((1 + D^2)^{-\frac{p}{2}}) < \infty\}$$

is called the *spectral dimension* of the spectral triple. We call $(A, \mathcal{H}, D)$ *θ-summable* if, for all $t > 0$,

$$\mathrm{Tr}(e^{-t(1+D^2)}) < \infty.$$

For spectral triples coming from non-unital C^*-algebras the definitions of summability are much more complex. See [40] for details. However, in the case we are interested in here, where the C^*-algebras are $S(X, \varphi, Q)$ and $U(X, \varphi, P)$, the definition simplifies to the following (since $S(X, \varphi, Q)$ has local units and $X^u(Q)$ is the unit space of the groupoid). For the stable algebra, $S(X, \varphi, Q)$, the spectral triple $(S, \mathcal{H}, D)$ is p-summable if, for all a in $C_c(X^u(Q))$,

$$\mathrm{Tr}(a(1 + D^2)^{-\frac{p}{2}}) < \infty$$

and it is θ-summable if, for all a in $C_c(X^u(Q))$ and for all $t > 0$,

$$\mathrm{Tr}(ae^{-t(1+D^2)}) < \infty.$$

The definitions are analogous for $U(X, \varphi, P)$ using b in $C_c(X^s(P))$.

5.2 Spectral Triples for Smale Spaces

We wish to construct spectral triples on the stable and unstable C^*-algebras associated with an irreducible Smale space, (X, d, φ), which are geometric and encode the dynamics in a natural way. Recall that we have represented both $S(X, \varphi, Q)$ and $U(X, \varphi, P)$ on the same Hilbert space $\mathcal{H} = \ell^2(X^h(P, Q))$ where P and Q are finite, mutually distinct, φ-invariant sets of periodic points, see Chapter 3.

We begin by constructing a function on $X^s(P)$ and use this function to define a spectral triple on $S(X, \varphi, Q)$. A completely analogous construction is available for constructing a spectral triple on $U(X, \varphi, P)$ using a function on $X^u(Q)$. We begin with the former.

Select $0 < \varepsilon \leq \lambda^{-1}\varepsilon_X/2$ where $\lambda > 0$ is the local expansion constant of the Smale space (X, d, φ). We aim to define a function $\omega_0 : X^s(P) \to [0, 1]$. Consider the closed sets $\overline{X^s(P, \varepsilon)}$ and $X^s(P) \setminus \varphi^{-1}(X^s(P, \varepsilon))$ and observe that these two sets are disjoint. Now an application of Urysohn's lemma implies that there exists a continuous function

$$\omega_0 : X^s(P) \to [0, 1]$$

such that $\omega_0(x) = 0$ for all $x \in \overline{X^s(P, \varepsilon)}$, and $\omega_0(x) = 1$ for all $x \in X^s(P) \setminus \varphi^{-1}(X^s(P, \varepsilon))$. We remark that in practice we may define ω_0 as desired on the complement of our two closed sets, but at this point we merely require that a continuous function exists. A typical function ω_0 is illustrated in Figure 5.1 on page 84, where the notation appearing in the figure is defined as follows.

Notation. For convenience of notation, we define the following sets. We note that the

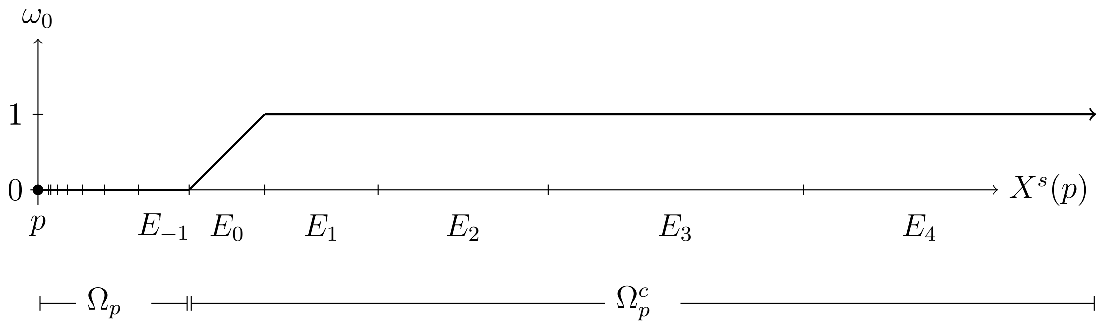


Figure 5.1: The function ω_0^p for some p in P

definitions anticipate the constructions in the sequel and are natural in that context.

$$\begin{aligned} \Omega_P &= X^s(P,\varepsilon) \setminus P, \\ \Omega_P^c &= X^s(P) \setminus X^s(P,\varepsilon), \\ E_0 &= \overline{\varphi^{-1}(X^s(P,\varepsilon)) \cap \Omega_P^c}, \\ E_N &= \varphi^{-N}(E_0). \end{aligned}$$

Let us make some remarks about these sets. First, observe that $\omega_0(x) = 0$ for $x \in \Omega_P$ and $\omega_0(x) \geq 0$ for $x \in \Omega_P^c$. Moreover, $\omega_0 : E_0 \to [0,1]$ is onto and continuous. Finally, we note that

$$\bigcup_{n \in \mathbb{Z}} E_N = X^s(P) \setminus P.$$

Using ω_0 allows us to encode the dynamics in the following way. Let x be a point in $X^s(P) \setminus P$. We aim to define a function which essentially counts the number of iterations it requires for x to be drawn into Ω_P if it begins in Ω_P^c and subtracts to number of inverse iterations it requires for x to be removed from Ω_P if it begins in Ω_P. To that end, define $\omega_s : X^s(P) \setminus P \to \mathbb{R}$ via

$$\omega_s(x) = \sum_{n=0}^{\infty} \omega_0 \circ \varphi^n(x) - \sum_{n=1}^{\infty} (1 - \omega_0) \circ \varphi^{-n}(x).$$

The function ω_s, arising from the function ω_0 in figure 5.1, is illustrated in Figure 5.2 on page 85. The following Lemma summarizes the essential properties of ω_s.

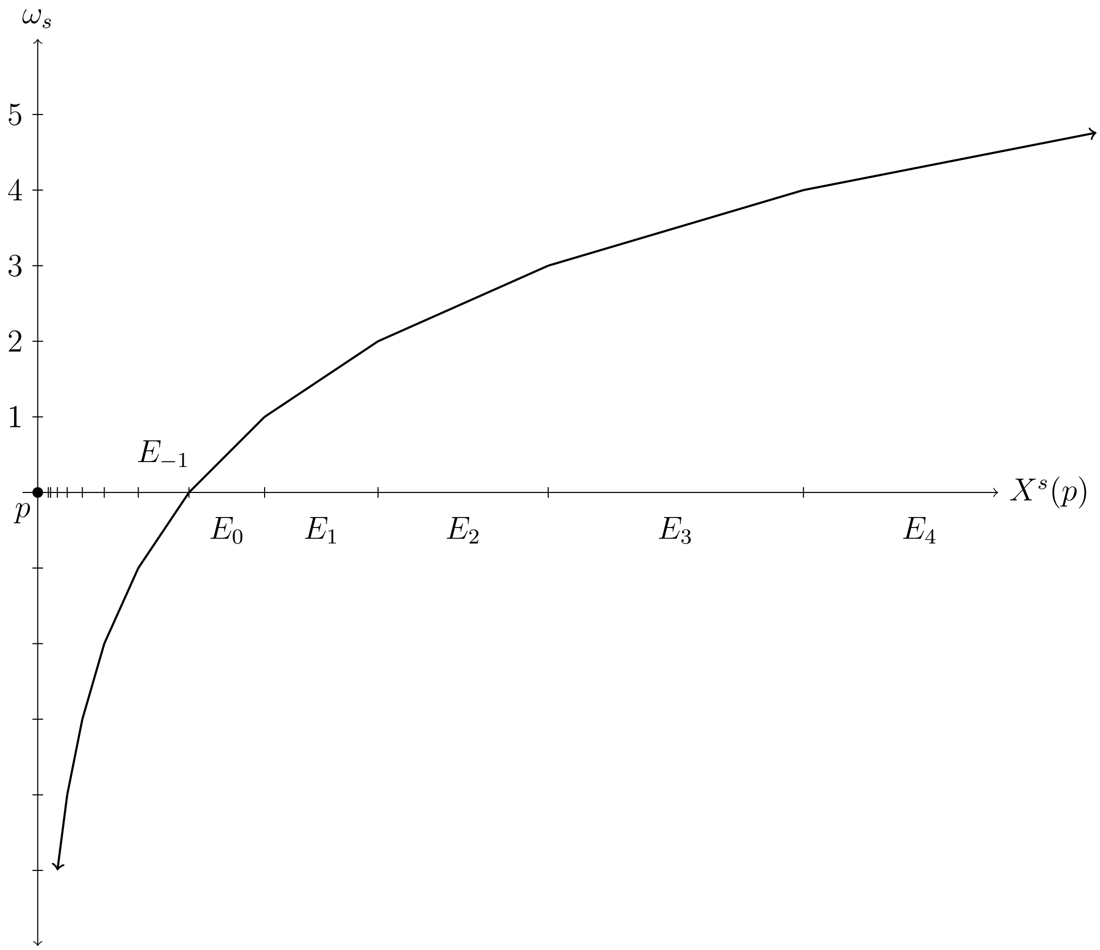


Figure 5.2: The function ω_s for some p in P

5.2.1 Lemma. *Suppose P is a finite, φ-invariant set of periodic points in a Smale space (X, d, φ) and $\omega_s : X^s(P) \setminus P \to \mathbb{R}$ is defined as above. Then,*

1. $\omega_s(x) \leq 0$ *for* $x \in \Omega_P$ *and* $\omega_s(x) \geq 0$ *for* $x \in \Omega_P^c$*,*
2. $\omega_s \circ \varphi - \omega_s = 1$*,*
3. $\omega_s(x) = \omega_0 \circ \varphi^N(x) + N$ *for* $x \in E_N$*, and*
4. ω_s *is continuous on* $X^s(P) \setminus P$*.*

Proof. First, suppose $x \in E_{-N}$ for some $N \in \mathbb{N}$. Then, the sum on the left, in the definition of ω_s, is zero since $\omega_0(x) = 0$ and $\varphi(x) \in E_{-N-1}$. The sum on the right is finite since $\varphi^N(x) \in E_0$ and $1 - \omega_0(\varphi^n(x)) = 0$ for all $n \geq N+1$. Moreover, we have the calculation

$$\omega_s(x) = -\sum_{n=1}^{\infty}(1-\omega_0)\circ\varphi^{-n}(x) = -(N-1) - (1-\omega_0)(\varphi^N(x)) = \omega_0(\varphi^N(x)) - N.$$

On the other hand, suppose $x \in E_N$ for some $N \in \mathbb{N} \cup 0$. Then, the sum on the right, in the definition of ω_s, is zero since $1 - \omega_0(x) = 0$ and $\varphi^{-1}(x) \in E_{N+1}$. The sum on the left is finite since $\varphi^N(x) \in E_0$ and $\omega_0(\varphi^n(x)) = 0$ for all $n \geq N+1$. Moreover, we have the calculation

$$\omega_s(x) = \sum_{n=0}^{\infty}\omega_0\circ\varphi^n(x) = \omega_0(\varphi^N(x)) + N.$$

This proves that ω_s is well-defined and the first three statements in the Lemma. For the fourth, we observe that $\omega_s(x)$ is continuous on E_N, for all $N \in \mathbb{Z}$, since ω_0 is continuous on E_0. Since $\bigcup_{N\in\mathbb{Z}} E_N = X^s(P)\setminus P$ it follows that ω_s is continuous. □

We now consider how the function ω_s interacts with functions in $C_c(G^s(X,\varphi,Q))$ supported on basic sets, see Lemma 3.2.5 for details on basic sets and for definitions of $Source(a)$ and $Range(a)$ for a in $S(X,\varphi,Q)$. We will use the following lemma extensively when proving that we have defined spectral triples.

5.2.2 Lemma. *Suppose P is a finite, φ-invariant set of periodic points in a Smale space (X,d,φ) and $\omega_s : X^s(P) \to \mathbb{R}$ is defined as above. Let $a \in C_c(G^s(X,\varphi,Q))$ be supported on a basic set $V^u(v,w,h^u,\delta)$ so that $Source(a) \subseteq X^u(w,\delta)$ and $Range(a) \subseteq X^u(v,\delta)$. Then,*

1. *there exists $N \in \mathbb{Z}$ such that $E_n \cap Source(a) = \emptyset$ for all $n \leq N$,*
2. *for all N, the number of points in $E_N \cap Source(a)$ is finite,*
3. *if $x \in E_N \cap Source(a)$, then there exists $K \in \mathbb{N}$ such that $h^u(x) \in \bigcup_{k=-K}^{K} E_{N+k}$.*

Proof. Since $Source(a)$ is a compact subset of $X^u(Q)$ and $P \notin X^u(Q)$, define $\delta_0 = \inf\{d(p,x) | p \in P, x \in Source(a)\} > 0$. Now there exists $N \in \mathbb{Z}$ such that $E_n \subset X^s(P,\delta_0)$ for all $n \leq N$. Therefore, $E_n \cap Source(a) = \emptyset$ for all $n \leq N$ as well. For the second claim,

$Source(a)$ and E_N are transverse and compact for all N, it follows that $E_N \cap Source(a)$ is finite.

For the third claim, we first note that given that a is supported in $V^u(v, w, h^u, \delta)$, there exists M such that for all $x \in Source(a)$, $d(\varphi^M(h^u(x)), \varphi^M(x)) < \varepsilon_X/2$ and $[\varphi^M(x), \varphi^M(h^u(x))] = \varphi^M(x)$. Moreover, it follows from Smale space axiom **C2** that there exists $L \in \mathbb{N}$ such that $D > \varepsilon_X$ where $D = \sup\{d(y, y') \mid y, y' \in E_L \text{ and } [y, y'] = y\}$; that is, we can find E_L so that E_L has diameter larger than ε_X on the stable set of each periodic point p in P. Now we claim that if $(h^u(x), x) \in V^u(v, w, h^u, \delta)$ has the property that $x \in E_m$ where $m \geq M + L + 1$, then $h^u(x) \in \bigcup_{k=-1}^{1} E_{m+k}$. Indeed, for all $(h^u(x), x)$ we have $d(\varphi^M(h^u(x)), \varphi^M(x)) < \varepsilon_X/2$ and if $\varphi^M(x)$ is in E_{L+1}, then by the triangle inequality we have $\varphi^M(h^u(x)) \in \cup_{k=-1}^{1} E_{L+1+k}$. Now applying φ^{-M} to $\varphi^M(x)$ and $\varphi^M(h^u(x))$ proves the claim. We are left with the case that $Source(a) \in E_n$ for $n < M + L + 1$. However, combining part (1) and (2) implies that there are only a finite number of such elements, so that we may define

$$K = \max\{1, |i - j| \mid h^u(x) \in E_i, x \in E_j, \text{ and } i, j < M + L + 1\},$$

which is finite. Now K has the property that if $x \in E_N \cap Source(a)$, then $h^u(x) \in \bigcup_{k=-K}^{K} E_{N+k}$. □

We will use ω_s to construct spectral triples on the stable algebra $S(X, \varphi, Q)$.

As mentioned in the introduction to this section, we also wish to construct a completely analogous function $\omega_u : X^u(Q) \to \mathbb{R}$. We will use this to define a spectral triple on the unstable algebra $U(X, \varphi, P)$.

5.2.1 A θ-Summable Spectral Triple

In this section, we define a spectral triple on each of the C^*-algebras $S(X, \varphi, Q)$ and $U(X, \varphi, P)$. As usual, we concentrate our attention on $S(X, \varphi, Q)$ and note that an analogous construction defines a spectral triple on $U(X, \varphi, P)$. Once we have shown that we have a spectral triple on $S(X, \varphi, Q)$, we extended it to a spectral triple on the

Ruelle algebra $S \rtimes_{\alpha_s} \mathbb{Z}$.

The idea is to use ω_s to count the number of iterations required for a point in $X^s(P)$ to either enter or be removed from Ω_P, depending on where in $X^s(P)$ it begins. Let us define an operator on $\mathcal{H} = \ell^2(X^h(P,Q))$ by

$$D_s\delta_x = \omega_s(x)\delta_x.$$

We note that the domain of D_s is given by

$$Domain(D_s) = \{\xi \mid \sum_{x\in X^h(P,Q)} \omega_s^2(x)|\xi(x)|^2 < \infty\}.$$

It is easy to see that D_s is symmetric by calculating

$$\begin{aligned} \langle D_s\delta_x, \delta_y\rangle &= \sum_{z\in X^h(P,Q)} D_s\delta_x(z)\delta_y(z) \\ &= \begin{cases} \omega_s(x)\delta_x & \text{if } x = y \\ 0 & \text{otherwise} \end{cases} \\ &= \sum_{z\in X^h(P,Q)} \delta_x(z)D_s\delta_y(z) \\ &= \langle \delta_x, D_s\delta_y\rangle . \end{aligned}$$

We would now like to show that $Domain(D_s) = Domain(D_s^*)$ from which it will follow that D_s is self-adjoint. Indeed,

$$Domain(D_s^*) \;=\; \{\eta \mid \xi \to \langle D_s\xi, \eta\rangle \text{ is bounded}\}$$

and $\langle D_s\xi, \eta\rangle = \langle \xi, \psi\rangle$ for some $\psi \in \ell^2(X^h(P,Q))$. Set $\xi = \delta_x$, and we compute

$$\begin{aligned} \langle D_s\xi, \eta\rangle &= \sum_{z\in X^h(P,Q)} \omega_s(z)\delta_x(z)\eta(z) = \omega_s(x)\eta(x) \\ \langle \xi, \psi\rangle &= \sum_{z\in X^h(P,Q)} \delta_x(z)\psi(z) = \psi(x). \end{aligned}$$

Thus, $\psi \in \ell^2(X^h(P,Q))$ implies that $\eta \in Domain(D_s)$. On the other hand, if $\eta \in$

$Domain(D_s)$, then

$$|\langle D_s\xi,\eta\rangle| = \left|\sum_{z\in X^h(P,Q)} \omega_s(z)\xi(z)\eta(z)\right| \leq \|\omega_s\xi\|_2\|\xi\|_2$$

which is bounded. Therefore, we have that $Domain(D_s) = Domain(D_s^*)$. Moreover, by the definition of the function ω_s, we observe that D_s is unbounded.

5.2.3 Lemma. *For $a \in C_c(G^s(X,\varphi,Q))$, the commutator $[D_s,a]$ is a bounded operator on $\mathcal{H}$.*

Proof. Let a in $C_c(G^s(X,\varphi,Q))$ be supported on a basic set $V^u(v,w,h^u,\delta)$. By part (3) of Lemma 5.2.2, if $x \in E_N \cap Source(a)$, then there exists $K \in \mathbb{N}$ such that $h^u(x) \in \bigcup_{k=-K}^{K} E_{N+k}$. Therefore, for any $x \in E_N \cap Source(a)$, using part (3) of lemma 5.2.1, we compute

$$\begin{aligned}
\|[D_s,a]\delta_x\| &= \|(\omega_s(h^u(x)) - \omega_s(x))a(h^u(x),x)\delta_{h^u(x)}\| \\
&= |\omega_s(h^u(x)) - \omega_s(x)||a(h^u(x),x)| \\
&\leq |\omega_0(\varphi^{N+K}(h^u(x))) + N + K - (\omega_0(\varphi^N(x)) + N)||a(h^u(x),x)| \\
&\leq (K+1)|a(h^u(x),x)|.
\end{aligned}$$

Since a is compactly supported, $|a(h^u(x),x)|$ attains a maximum value. Moreover, everything above is independent of N so that $[D_s,a]$ is bounded. For the general case we recall that any element of $C_c(G^s(X,\varphi,Q))$ is in the span of functions supported on basic sets. □

5.2.4 Proposition. *For every a in $S(X,\varphi,Q)$, the operator $a(1+D_s^2)^{-1}$ is compact on $\mathcal{H}$.*

Proof. Let a_0 in $S(X,\varphi,Q)$ be supported on a basic set of the form $V^u(v,w,h^u,\delta)$. By part (1) of Lemma 5.2.2, there exists M such that $E_m \cap Source(a_0) = \emptyset$ for all $m \leq M$. Furthermore, by part (2) of Lemma 5.2.2, the number of elements in $E_N \cap Source(a_0)$ is finite for all N. Now using part (3) of Lemma 5.2.1, for $x \in E_N \cap Source(a_0)$ we have

$$\|a_0(1+D_s^2)^{-1}\delta_x\| = \|\frac{a_0(h^u(x),x)}{1+\omega_s^2(x)}\delta_h^u(x)\| \leq \frac{|a_0(h^u(x),x)|}{1+N^2}.$$

Since a_0 has compact support, let $A = \sup\{|a_0(h^u(x), x)| \mid x \in \mathit{Source}(a_0)\}$. Moreover since h^u takes basis vectors to basis vectors and is a homeomorphism from $\mathit{Source}(a_0)$ to $\mathit{Range}(a_0)$ it follows that, restricted to E_N,

$$\|a_0(1 + D_s^2)^{-1}\| \leq \frac{A}{1 + N^2}.$$

Therefore, $a_0(1+D_s^2)^{-1}$ is a norm limit of finite rank operators. Moreover, a in $S(X, \varphi, Q)$ is a norm limit of finite sums of operators of the form a_0, so $a(1 + D_s^2)^{-1}$ is compact as well, since the compact operators are closed. □

Before arriving at our main theorem for the section, we must delve into a technical result about topological entropy. For an irreducible Smale space (X, d, φ), recall, from Section 2.1, that the topological entropy of (X, d, φ) is denoted $h(X, \varphi)$ and is the exponential growth rate of the number of essentially different orbit segments of length N. We state the following result which is obtained by combining Lemma 5.9 and Proposition 5.12 in Brady Killough's dissertation [28]. There are also several similar results appearing in [30].

5.2.5 Theorem ([28])**.** *Suppose (X, d, φ) is an irreducible Smale space with P and Q distinct, finite, φ-invariant sets of periodic points. Then for any $\delta_0, \delta_1 > 0$ and any $w \in X^u(Q)$, $\#\{x \mid \varphi^N(X^s(P, \delta_0) \cap X^u(w, \delta_1)\}$ is finite. Moreover,*

$$\lim_{N\to\infty} |\frac{1}{N} \log(\#\{x \mid \varphi^N(X^s(P, \delta_0) \cap X^u(w, \delta_1)\}) - h(X, \varphi)| = 0.$$

5.2.6 Theorem. *$(S, \mathcal{H}, D_s)$ is a non-unital, θ-summable spectral triple.*

Proof. We have shown that $(S, \mathcal{H}, D_s)$ is a spectral triple. It remains to show that $(S, \mathcal{H}, D_s)$ is θ-summable. We must show that, for a in $C_c(X^u(Q))$ a positive operator, we have $\mathrm{Tr}(ae^{-t(1+D^2)}) < \infty$ for all $t > 0$. By part (1) of Lemma 5.2.2, there exists M such that $E_m \cap \mathit{Source}(a) = \emptyset$ for all $m \leq M$. Furthermore, by part (2) of Lemma 5.2.2, the number of elements in $E_N \cap \mathit{Source}(a)$ is finite for all N. Now using part (3) of

Lemma 5.2.1, for $x \in E_N \cap Source(a)$ we have

$$\begin{aligned} \mathrm{Tr}(ae^{-t(1+D^2)}) &= \sum_{x \in X^h(P,Q)} \left\langle ae^{-t(1+D^2)}\delta_x, \delta_x \right\rangle \\ &= \sum_{x \in Source(a)} \frac{a(x,x)}{e^{t(1+\omega_s^2(x))}} \\ &\leq \sum_{n=M}^{\infty} \left(\sum_{\#\{x \mid x \in E_n \cap Source(a)\}} \frac{a(x,x)}{e^{t(1+(n)^2)}} \right). \end{aligned} \tag{5.1}$$

Now from Theorem 5.2.5, for $\varepsilon > 0$, there exists N such that for all $n \geq N$,

$$\#\{x \mid x \in E_n \cap Source(a)\} < e^{n(h(X,\varphi)+\varepsilon)}.$$

Therefore, we have

$$\begin{aligned} \sum_{\#\{x \mid x \in E_n \cap Source(a)\}} \frac{a(x,x)}{e^{t(1+(n)^2)}} &< \frac{a(x,x)e^{n(h(X,\varphi)+\varepsilon)}}{e^{t(1+(n)^2)}} \\ &= a(x,x)e^{n(h(X,\varphi)+\varepsilon)-t(1+(n)^2)} \\ &< a(x,x)e^{n(h(X,\varphi)+\varepsilon)-tn^2} \\ &= a(x,x)e^{n(h(X,\varphi)+\varepsilon-tn)}. \end{aligned}$$

Putting this into (5.1) and letting R denote the first $N-1$ terms of the sum yields

$$\mathrm{Tr}(ae^{-t(1+D^2)}) = R + \sum_{n=M}^{\infty} a(x,x)e^{n(h(X,\varphi)+\varepsilon-tn)},$$

which converges since we can choose N sufficiently large that $tN > h(X,\varphi) + \varepsilon$. □

Finally, we show that we can extend D_s to the Ruelle algebra $S \rtimes_{\alpha_s} \mathbb{Z}$.

5.2.7 Theorem. *$(S \rtimes_{\alpha_s} \mathbb{Z}, \mathcal{H}, D_s)$ is a non-unital, θ-summable spectral triple.*

Proof. Note that $S \rtimes_{\alpha_s} \mathbb{Z}$ is represented on $\mathcal{H}$ and $\mathrm{span}\{a \cdot u^n \mid a \in S(X,\varphi,Q)\}$ is dense in $S \rtimes_{\alpha_s} \mathbb{Z}$. Therefore, we need only show that

$$[a \cdot u^n, D_s] = a[u^n, D_s] + [a, D_s]u^n$$

is a bounded operator on $\mathcal{H}$ which amounts to showing that $[u, D_s] \in \mathcal{B}(\mathcal{H})$. By part (2) of Lemma 5.2.1 we have $[u, D_s] = u$ so that $\|[u, D_s]\| = 1$. $\square$

In a similar fashion, with $D_u\delta_x = \omega_u(x)\delta_x$ we obtain a spectral triple $(U, \mathcal{H}, D_u)$ which also extends to a spectral triple $(U \rtimes_{\alpha_u} \mathbb{Z}, \mathcal{H}, D_u)$ on the unstable Ruelle algebra.

5.2.2 A p-Summable Spectral Triple

We use special cases of the functions ω_s and ω_u to define spectral triples on each of the C^*-algebras $S(X, \varphi, Q)$ and $U(X, \varphi, P)$. As usual, we concentrate our attention on $S(X, \varphi, Q)$ and observe that an analogous construction defines a spectral triple on (U, φ, P).

Let us define ω_0 to be locally Lipschitz continuous; that is, there exists a constant C_0 such that if $x, y \in E_0$, $[x, y] = x$, and $d(x, y) < \varepsilon_X/2$, then

$$|\omega_0(x) - \omega_0(y)| < C_0 d(x, y)$$

where the metric comes from the Smale space itself. In fact, since (X, d) is a compact metric space we can always define such a function using the metric and regarding E_0 as a disjoint union of closed sets, one for each element of P. Let us also define a constant $C_s = 2KC_0$ where

$$K = \max\{k > 0 \mid [x, y] = x, d(x, y) < \varepsilon_X/2, \text{ with } x \in E_0 \text{ and } y \in E_k\}.$$

5.2.8 Lemma. *The function ω_s is locally Lipschitz continuous on $\cup_{n=0}^{\infty} E_n$; that is, if $x, y \in \cup_{n=0}^{\infty} E_n$, $d(x, y) < \varepsilon_X/2$ and $[x, y] = x$, then*

$$|\omega_s(x) - \omega_s(y)| < C_s d(x, y).$$

Proof. First observe that $\omega_s(x)$ is locally Lipschitz continuous, with Lipschitz constant C_0, on E_N, for all $N \in \mathbb{N}$. Indeed, suppose $x, y \in E_N$ such that $[x, y] = x$. Then, using

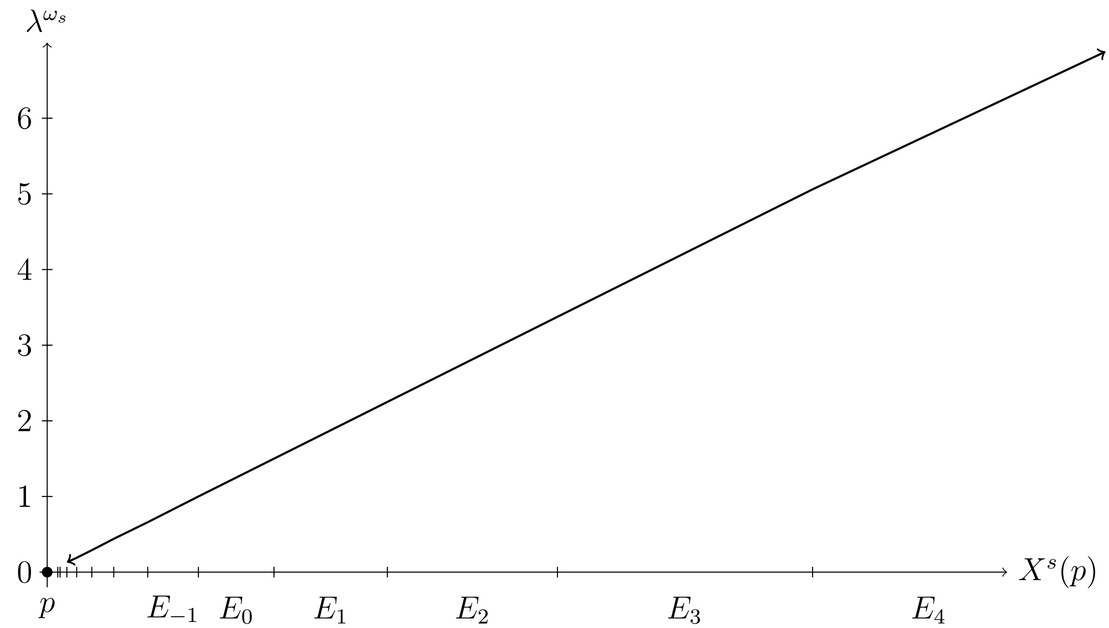


Figure 5.3: The function λ^{ω_s} for some p in P

part (3) of Lemma 5.2.1,

$$\begin{aligned} |\omega_s(x) - \omega_s(y)| &= |(\omega_0(\varphi^N(x)) + N) - (\omega_0(\varphi^N(y)) + N)| \\ &= |\omega_0(\varphi^N(x)) - \omega_0(\varphi^N(y))| \\ &< C_0 d(\varphi^N(x), \varphi^N(y)) < C_0 \lambda^{-N} d(x,y). \end{aligned}$$

Now suppose $x, y \in \cup_{n=0}^{\infty} E_n$, $d(x,y) < \varepsilon_X/2$ and $[x,y] = x$. Then we note that if $x \in E_N$ then $y \in \cup_{k=-K+N}^{K+N} E_k$ where K comes from the definition of C_s. Now the triangle inequality gives the result. □

The function λ^{ω_s}, arising from the function ω_s in figure 5.2, is illustrated in Figure 5.3 on page 93.

Define an operator $\mathfrak{D}_s$ on $\mathcal{H} = \ell^2(P, Q)$ via

$$\mathfrak{D}_s \delta_x = \lambda^{\omega_s(x)} \delta_x$$

where $\lambda > 1$ the local growth rate of (X, d, φ), see bracket axioms **C1** and **C2** in section

2.2. In a similar manner to the previous section, $\mathfrak{D}_s$ is self-adjoint, unbounded, and has dense domain.

5.2.9 Lemma. *For a in $C_c(G^s(X,\varphi,Q))$, the commutator $[a,\mathfrak{D}_s]$ is a bounded operator on $\mathcal{H}$.*

Proof. Let a in $C_c(G^s(X,\varphi,Q))$ be supported on a basic set $V^u(v,w,h^u,\delta)$. Recall that there exists M such that, for $x \in Source(a)$, we have $d(h^u(x),x) < \varepsilon_X/2$. By part (1), (2), and (3) of Lemma 5.2.2, we can find $K \geq M$ such that both $Source(a)$ and $Range(a)$ are in $\cup_{k=K}^{\infty} E_k$ and the number of elements in $Source(a) \cap (X^s(P) \setminus \cup_{k=K}^{\infty} E_k)$ is finite.

If $x \in Source(a) \cap E_k$, for $k \geq K$, then we compute

$$\begin{aligned}
\|[a,\mathfrak{D}_s]\delta_x\| &= \|(\lambda^{\omega_s(x)} - \lambda^{\omega_s(h^u(x))})a(h^u(x),x)\delta_{h^u(x)}\| \\
&= |\lambda^{\omega_s(x)} - \lambda^{\omega_s(h^u(x))}||a(h^u(x),x)| \\
&= \lambda^{k+\omega_0(\varphi^k(x))}|1 - \lambda^{(k-K)+\omega_s(\varphi^{(k-K)}(h^u(x)))-(k-K)-\omega_s(\varphi^{(k-K)}(x))}||a(h^u(x),x)| \\
&\leq \lambda^{k+1}|1 - \lambda^{\omega_s(\varphi^{(k-K)}(h^u(x)))-\omega_s(\varphi^{(k-K)}(x)))}||a(h^u(x),x)| \\
&\leq \lambda^{k+1}|1 - \lambda^{C_s d(\varphi^{(k-K)}(h^u(x)),\varphi^{(k-K)}(x))}||a(h^u(x),x)| \\
&\leq \lambda^{k+1}|1 - \lambda^{C_s\lambda^{-(k-(K-M))}\varepsilon_X/2}||a(h^u(x),x)| \\
&< \lambda^{k+1}|1 - \lambda^{\log_\lambda(1+C_s\lambda^{K-M-k}\varepsilon_X/2)}||a(h^u(x),x)| \\
&= \lambda^{k+1}|1 - (1+C_s\lambda^{K-M-k}\varepsilon_X/2)||a(h^u(x),x)| \\
&= \lambda^{k+1}C_s\lambda^{K-M-k}\varepsilon_X/2|a(h^u(x),x)| \\
&= C_s\lambda^{K-M+1}\varepsilon_X/2|a(h^u(x),x)|
\end{aligned}$$

where K and M depend only on the function a and since a is compactly supported attains its maximum. Thus, in this case $[a,\mathfrak{D}_s]$ is bounded.

On the other hand, if $x \in Source(a) \cap (X^s(P) \setminus \cup_{k=K}^{\infty} E_k)$ then there are only a finite number of elements so that we can take the maximum value of $\|[a,\mathfrak{D}_s]\delta_x\|$. So $[a,\mathfrak{D}_s]$ is bounded and the Lemma is proven. □

To complete the proof that $(S,\mathcal{H},\mathfrak{D}_s)$ is a non-unital spectral triple we need only show that $a(1+\mathfrak{D}_s^2)^{-1}$ is a compact operator for every a in $S(X,\varphi,Q)$. The same argument as presented in Section 5.2.1 gives the result. We will now show that $(S, \mathcal{H}, \mathfrak{D}_s)$ is a finitely summable spectral triple. Indeed, $(S, \mathcal{H}, \mathfrak{D}_s)$ is $\log_\lambda(e)h(X,\varphi)$-summable, where $h(X,\varphi)$ is the topological entropy of the Smale space (X, d, φ). We note that the factor $\log_\lambda(e)$ is merely a base change from a base e logarithm to a base λ logarithm.

5.2.10 Theorem. *$(S, \mathcal{H}, \mathfrak{D}_s)$ is a non-unital, $\log_\lambda(e)h(X,\varphi)$-summable spectral triple, where $h(X,\varphi)$ is the topological entropy of the Smale space (X, d, φ).*

Proof. We have shown that $(S, \mathcal{H}, \mathfrak{D}_s)$ is a spectral triple. It remains to show that $(S, \mathcal{H}, \mathfrak{D}_s)$ is summable. We must show that, for a in $C_c(X^u(Q))$ a positive operator, we have

$$\mathrm{Tr}(a(1+\mathfrak{D}^2)^{-\frac{s}{2}}) < \infty$$

for some $s > 0$. By part (1) of Lemma 5.2.2, there exists M such that $E_m \cap Source(a) = \emptyset$ for all $m \leq M$. Furthermore, by part (2) of Lemma 5.2.2, the number of elements in $E_N \cap Source(a)$ is finite for all N. Now using part (3) of Lemma 5.2.1, for $x \in E_N \cap Source(a)$ we have

$$\begin{aligned} \mathrm{Tr}(a(1+\mathfrak{D}^2)^{-\frac{s}{2}}) &= \sum_{x \in X^h(P,Q)} \left\langle a(1+\mathfrak{D}^2)^{-\frac{s}{2}})\delta_x, \delta_x \right\rangle \\ &= \sum_{x \in Source(a)} \frac{a(x,x)}{(1+\lambda^{2\omega_s(x)})^{s/2}} \\ &\leq \sum_{n=M}^{\infty} \left(\sum_{\#\{x \,|\, x \in E_n \cap Source(a)\}} \frac{a(x,x)}{(1+\lambda^{2n})^{s/2}} \right). \end{aligned} \tag{5.2}$$

Now from Theorem 5.2.5, for $\varepsilon > 0$, there exists N such that for all $n \geq N$,

$$e^{n(h(X,\varphi)-\varepsilon)} < \#\{x \mid x \in E_n \cap Source(a)\} < e^{n(h(X,\varphi)+\varepsilon)}. \tag{5.3}$$

Therefore, we have

$$\begin{aligned}\sum_{\#\{x \,|\, x \in E_n \cap Source(a)\}} \frac{a(x,x)}{(1+\lambda^{2n})^{s/2}} &< \frac{a(x,x)e^{n(h(X,\varphi)+\varepsilon)}}{(1+\lambda^{2n})^{s/2}} \\ &< \frac{a(x,x)e^{n(h(X,\varphi)+\varepsilon)}}{(\lambda^{2n})^{s/2}} \\ &= \frac{a(x,x)\lambda^{\log_\lambda(e)n(h(X,\varphi)+\varepsilon)}}{(\lambda^{sn})} \\ &= a(x,x)\lambda^{n(\log_\lambda(e)h(X,\varphi)+\log_\lambda(e)\varepsilon)-sn} \\ &= a(x,x)\lambda^{n((\log_\lambda(e)h(X,\varphi)+\log_\lambda(e)\varepsilon)-s)}\end{aligned}$$

Putting this into (5.2) and letting R denote the first $N-1$ terms of the sum yields

$$\mathrm{Tr}(a(1+\mathfrak{D}^2)^{-\frac{s}{2}}) \quad < \quad R + \sum_{n=M}^{\infty} a(x,x)\lambda^{n((\log_\lambda(e)h(X,\varphi)+\log_\lambda(e)\varepsilon)-s)},$$

which converges geometrically for $s > \log_\lambda(e)h(X,\varphi)+\log_\lambda(e)\varepsilon$. Since this holds for any $\varepsilon > 0$ we have that

$$\log_\lambda(e)h(X,\varphi) \geq \inf\{s \mid \mathrm{Tr}(a(1+\mathfrak{D}^2)^{-\frac{s}{2}}) < \infty\}.$$

On the other hand, using the other inequality in (5.3), a similar computation shows that

$$\mathrm{Tr}(a(1+\mathfrak{D}^2)^{-\frac{s}{2}}) > R + \sum_{n=M}^{\infty} \frac{a(x,x)\lambda^s}{2}\lambda^{n((\log_\lambda(e)h(X,\varphi)-\log_\lambda(e)\varepsilon)-s)},$$

which converges geometrically only if $s > \log_\lambda(e)h(X,\varphi) - \log_\lambda(e)\varepsilon$ for every $\varepsilon > 0$. Therefore, we have

$$\log_\lambda(e)h(X,\varphi) = \inf\{s \mid \mathrm{Tr}(a(1+\mathfrak{D}^2)^{-\frac{s}{2}}) < \infty\}.$$

□

Of course we also obtain a $\log_\lambda(e)h(X,\varphi)$-summable spectral triple on $U(X,\varphi,P)$ in an analogous fashion.

Chapter 6

Fredholm Index for a Shift of Finite Type

6.1 A Fredholm Module for the C^*-algebras Associated with a Shift of Finite Type

Let (X_A, φ_A) be a shift of finite type associated with a non-negative $K \times K$ integer matrix A which is irreducible, see section 2.3.1 for details. We wish to construct a Fredholm module on the stable and unstable C^*-algebras associated with (X_A, φ_A); $S(X_A, \varphi_A, Q)$ and $U(X_A, \varphi_A, P)$ where both P and Q are disjoint φ_A-invariant sets of periodic points. Recall that we have represented both C^*-algebras on the same Hilbert space $\mathcal{H} = \ell^2(X^h(P,Q))$. We note that for a shift of finite type, this is saying that every point in $X^h(P,Q)$ is a path in the directed graph G_A, coming from A, that is left tail equivalent to a periodic point in Q and right tail equivalent to a periodic point in P.

We will work exclusively with the stable algebra of a shift of finite type, which we denote by $S(X_A, \varphi_A, Q)$, and the associated Ruelle algebra, $S \rtimes_{\alpha_s} \mathbb{Z}$. We aim to construct a φ_A-invariant projection on $\mathcal{H} = \ell^2(X^h(P,Q))$. To this end, specify a 'special edge' $e \neq p_0$, from the graph G_A, such that $i(p_1) = t(e)$ where $p = \cdots p_{-2}p_{-1}.p_0p_1p_2\cdots$ is a periodic point in P. Define

$$E_0 = \{x \in \mathcal{H} | x = \cdots x_{-2}x_{-1}.ep_1p_2 \ldots \text{ such that } p \in P\}$$

and $E_N = \varphi^{-N}(E_0)$. Finally let $E = \bigcup_{N \in \mathbb{Z}} E_N$ and observe that a point in $\mathcal{H}$ is in E if it is right tail equivalent to a periodic point in P, and the edge exactly preceding the periodic right tail is the edge e. Note that E is φ_A invariant.

Using E, we define a projection $\mathscr{P}_e$ on $\mathcal{H} = \ell^2(X^h(P,Q))$ via

$$\mathscr{P}_e \delta_x = \begin{cases} \delta_x & \text{if } x \notin E \\ 0 & \text{if } x \in E \end{cases}$$

6.1.1 Lemma. *For $a \in S(X_A, \varphi_A, Q)$, we have $[\mathscr{P}_e, a] \in \mathcal{K}(\mathcal{H})$.*

Proof. Let a in $S(X_A, \varphi_A, Q)$ be supported on a basic set of the form $V^u(v, w, h^u, \delta)$. We observe that for any point $(h^u(x), x)$ in $V^u(v, w, h^u, \delta)$, the definition of the local homeomorphism h^u implies that $d(\varphi^N(h^u(x)), \varphi^N(x)) < \varepsilon_X/2$ so that a typical point in $V^u(v, w, h^u, \delta)$ is a pair $(h^u(x), x)$ with

$$\begin{aligned} x &= \cdots w_{-1}.w_0w_1 \cdots w_N x_{N+1} x_{N+2} \cdots \\ h^u(x) &= \cdots v_{-1}.v_0v_1 \cdots v_N x_{N+1} x_{N+2} \cdots \end{aligned}$$

See lemma 3.1.2.

We shall use such points to compute the commutator:

$$\begin{aligned} a\mathscr{P}_e\delta_x &= \begin{cases} a(h^u(x), x)\delta_{h^u(x)} & \text{if } x \notin E \\ 0 & \text{if } x \in E \text{ or } x \notin \mathit{Source}(a) \end{cases} \\ \mathscr{P}_e a\delta_x &= \begin{cases} a(h^u(x), x)\delta_{h^u(x)} & \text{if } h^u(x) \notin E \\ 0 & \text{if } h^u(x) \in E \text{ or } h^u(x) \notin \mathit{Range}(a) \end{cases} \end{aligned}$$

Thus, we have

$$[\mathscr{P}_e, a]\delta_x = \begin{cases} a(h^u(x), x)\delta_{h^u(x)} & \text{if } h^u(x) \notin E, x \in E, \text{ and } (h^u(x), x) \in supp(a) \\ -a(h^u(x), x)\delta_{h^u(x)} & \text{if } h^u(x) \in E, x \notin E, \text{ and } (h^u(x), x) \in supp(a) \\ 0 & \text{if } (h^u(x), x) \notin supp(a) \\ 0 & \text{if } h^u(x) \in E \text{ and } x \in E \\ 0 & \text{if } h^u(x) \notin E \text{ and } x \notin E \end{cases}$$

At this point, we need only compute the number of points x such that $[\mathscr{P}_e, a]\delta_x$ is non-zero. We notice immediately that if either of x or $h^u(x)$ are in E_M for $M > N$ then both x and $h^u(x)$ are in E. So for $[\mathscr{P}_e, a]\delta_x$ to be non-zero we must have that either x or $h^u(x)$ in E_M for some $M \leq N$ but not both. However, this implies that there is at most one of either x or $h^u(x)$ in E_M since we must have $x_{N+1}x_{N+2}\cdots = p_{N+1}p_{N+2}\cdots$ and both v and w are fixed in the support of a. □

Given a projection $\mathscr{P}_e$ such that $[\mathscr{P}_e, a] \in \mathcal{K}(\mathcal{H})$ for all a in $S(X_A, \varphi_A, Q)$, algebraic arguments imply that $(S, \mathcal{H}, F_e)$ is a Fredholm module where $F_e = 2\mathscr{P}_e - 1$, see Example 8.1.7 in [24]. Note that $F_e : \mathcal{H} \to \mathcal{H}$ is given by

$$F_e\delta_x = \begin{cases} \delta_x & \text{if } x \notin E \\ -\delta_x & \text{if } x \in E \end{cases}$$

We have therefore proven the following.

6.1.2 Proposition. *The triple $(S, \mathcal{H}, F_e)$ is a Fredholm module where $F_e = 2\mathscr{P}_e - 1$.*

At this point, we aim to extend the Fredholm module to the Ruelle algebra $S \rtimes_{\alpha_s} \mathbb{Z}$. Indeed, as in Section 5.2.1, we have $S \rtimes_{\alpha_s} \mathbb{Z}$ is represented on $\mathcal{H}$ and $\text{span}\{a \cdot u^n \mid a \in S(X_A, \varphi_A, Q)\}$ is dense in $S \rtimes_{\alpha_s} \mathbb{Z}$. Therefore, we need only show that

$$[a \cdot u^n, \mathscr{P}_e] = a[u, \mathscr{P}_e] + [a, \mathscr{P}_e]u$$

is a compact operator on $\mathcal{H}$, which amounts to showing that $[u, \mathscr{P}_e] \in \mathcal{K}(\mathcal{H})$. However, E is φ_A-invariant so that $[u, \mathscr{P}_e] = 0$. Whence, we have shown:

6.1.3 Proposition. *The triple $(S \rtimes_{\alpha_s} \mathbb{Z}, \mathcal{H}, F_e)$ is a Fredholm module where $F_e = 2\mathscr{P}_e - 1$.*

6.2 Unitaries in $S \rtimes_{\alpha_s} \mathbb{Z}$

At this point, we take a small detour to consider unitaries in the unitization of the crossed product $S \rtimes_{\alpha_s} \mathbb{Z}$. We are aiming to compute an index pairing of unitary operators in $(S \rtimes_{\alpha_s} \mathbb{Z})^{\sim}$ with the Fredholm module $(S \rtimes_{\alpha_s} \mathbb{Z}, \mathcal{H}, F_e)$ defined in the Proposition 6.1.3. Our aim in the present section is to geometrically define these unitaries.

First, we note that for a shift of finite type, $S(X_A, \varphi_A, Q)$ is an AF-algebra so that $K_1(S) = 0$, see [35]. Therefore, the Pimsner -Voiculescu sequence [33] reduces to the following exact sequence:

$$0 \longrightarrow K_1(S \rtimes_{\alpha_s} \mathbb{Z}) \longrightarrow K_0(S) \xrightarrow{id - \alpha_*} K_0(S) \longrightarrow K_0(S \rtimes_{\alpha_s} \mathbb{Z}) \longrightarrow 0.$$

Whence, $K_1(S \rtimes_{\alpha_s} \mathbb{Z}) \cong ker(id - \alpha_*)$ where $id - \alpha_* : K_0(S) \to K_0(S)$. Supposing that A is the adjacency matrix of our shift of finite type, we have

$$K_1(S \rtimes_{\alpha_s} \mathbb{Z}) \;\; \cong \;\; \{\Upsilon \in \mathbb{Z}^K | \Upsilon(A - I) = 0\}$$

where Υ is interpreted as a row vector. Let $\Upsilon = \Upsilon^+ - \Upsilon^-$ where both Υ^+ and Υ^- hace non-negative entries. Now $\Upsilon(A - I) = 0$ implies that $\Upsilon^+ A + \Upsilon^- I = \Upsilon^- A + \Upsilon^+ I$. We will use this formula to define a partial isometry v. First, notice that Υ^+ and Υ^- are column vectors and each entry corresponds with a vertex v_i in G_A. Consider all truncated edge paths in X_G given by

$$\Xi_{0,v_i} = \{\cdots q_{-2} q_{-1} . \xi_0 \;\mid\; q \in Q, q_0 \neq \xi_0, t(q_0) = i(\xi_0), \text{ and } t(\xi_0) = v_i\}$$

and for each $\xi \in \Xi_{0,v_i}$ define the cylinder set

$$V_{0,v_i}(\xi) = \{x \in X_A \;\mid\; x_i = \xi_i \text{ for all } i \leq 0\}.$$

That is, given an edge ξ_0, with terminal vertex v_i, in the graph G, elements of $V_{0,v_i}(\xi)$ consist of elements of $x \in \mathcal{H} = \ell^2(P, Q)$ of the form

$$\cdots x_{-3} x_{-2} x_{-1} . x_0 x_1 x_2 x_3 \cdots = q_{-3} q_{-2} q_{-1} . \xi_0 * * * \cdots ,$$

where $t(x_i) = i(x_{i+1})$, $q = \cdots q_{-2}q_{-1}.q_0q_1\cdots$ is a periodic point in Q, and $*$ indicates that we may fill in the corresponding entry with any allowable edge in the graph G. Given a cylinder set $\eta = \cdots \eta_{-2}\eta_{-1}.\eta_0\eta_1\cdots\eta_n * * \cdots$, we introduce the notation $\eta(x) = \cdots \eta_{-2}\eta_{-1}.\eta_0\eta_1\cdots\eta_n x_{n+1}x_{n+2}\cdots$ where we have filled in the $*'s$ with the right tail of a point $x \in X^s(P)$.

Suppose that we have cylinder sets $\eta(x) = \cdots\eta_{-1}.\eta_0\eta_1\cdots\eta_n * * \cdots$ and $\eta'(x) = \cdots\eta'_{-1}.\eta'_0\eta'_1\cdots\eta'_n * * \cdots$, then define $e(\eta',\eta)$ to be the partial isometry in $S(X_A, \varphi_A, Q)$ defined, for x in $\mathcal{H}$, by

$$e(\eta',\eta)\delta_x = \begin{cases} \delta_{\eta'(x)} & \text{if } x \in \eta \\ 0 & \text{otherwise} \end{cases}$$

Now, using the column vectors Υ^+ and Υ^-, we define projections

$$\begin{aligned} p^+ &= \sum_{i=1}^{N}\sum_{\#\Upsilon_i^+}\{e(y^+,y^+)|y^+ \in V_{0,v_i}(\xi)\} \\ p^- &= \sum_{i=1}^{N}\sum_{\#\Upsilon_i^-}\{e(y^-,y^-)|y^- \in V_{0,v_i}(\xi)\}, \end{aligned}$$

Note that y^+ is a cylinder set in $V_{0,v_i}(\xi)$. Now for each $y^+ \in V_{0,v_i}(\xi)$, define the union of cylinder sets

$$\varphi(y^+)A = \bigcup_{\{a\in E\,|\,i(a)=v_i\}} \{x \in \mathcal{H} \mid x_i = y^+_{i+1} \text{ for all } i \leq -1 \text{ and } x_0 = a\}.$$

Similarly, for each $y^- \in V_{0,v_i}(\xi)$, define the union of cylinder sets

$$\varphi(y^-)A = \bigcup_{\{b\in E\,|\,i(b)=v_i\}} \{x \in \mathcal{H} \mid x_i = y^-_{i+1} \text{ for all } i \leq -1 \text{ and } x_0 = b\}.$$

Now we have sets $\Omega = \{\varphi(y^+)A\} \cup \{y^-\}$, which corresponds with $\Upsilon^+A + \Upsilon^-I$, and $\Omega' = \{y^+\}\cup\{\varphi(y^-)A\}$, which corresponds with $\Upsilon^-A+\Upsilon^+I$. The fact that $\Upsilon^+A+\Upsilon^-I = \Upsilon^-A+\Upsilon^+I$ gives rise to a bijection $\psi : \Omega \to \Omega'$:

$$\begin{array}{ccc} \Omega & & \Omega' \\ \varphi(y^+)A & \overset{\psi}{\longrightarrow} & (y^+) \\ (y^-) & & \varphi(y^-)A \end{array}$$

That is, there is a bijective correspondence between paths in Ω and paths in Ω' that begin and end on the same vertex. We obtain a partial isometry

$$v = \sum_{x \text{ a path in } \Omega} e(\psi(x), x)$$

and we note that the initial projection is $v^*v = \mathcal{X}_\Omega$ and the range projection is $vv^* = \mathcal{X}_{\Omega'}$.

We claim that the operator

$$z = (up^+ + p^-u^*)v - (1 - \mathcal{X}_\Omega)$$

is a unitary in $(S \rtimes_{\alpha_s} \mathbb{Z})^\sim$. Indeed,

$$\begin{aligned} z^*z &= (v^*(p^+u^* + up^-) - (1 - \mathcal{X}_\Omega))((up^+ + p^-u^*)v - (1 - \mathcal{X}_\Omega)) \\ &= v^*(p^+u^*up^+ + p^+u^*p^-u^* + up^-up^+ + up^-p^-u^*)v + (1 - \mathcal{X}_\Omega)^2 \\ &= v^*(p^+ + \alpha(p^-))v + (1 - \mathcal{X}_\Omega) \\ &= v^*(\mathcal{X}_{\Omega'})v + (1 - \mathcal{X}_\Omega) \\ &= \mathcal{X}_\Omega + 1 - \mathcal{X}_\Omega = 1 \\ zz^* &= ((up^+ + p^-u^*)v - (1 - \mathcal{X}_\Omega))(v^*(p^+u^* + up^-) - (1 - \mathcal{X}_\Omega)) \\ &= (up^+ + p^-u^*)vv^*(p^+u^* + up^-) + (1 - \mathcal{X}_\Omega)^2 \\ &= (up^+ + p^-u^*)(p^+ + up^-u^*)(p^+u^* + up^-) + (1 - \mathcal{X}_\Omega) \\ &= (up^+ + p^-u^*)(p^+u^* + up^-) + (1 - \mathcal{X}_\Omega) \\ &= (up^+u^* + p^-) + (1 - \mathcal{X}_\Omega) \\ &= \mathcal{X}_\Omega + 1 - \mathcal{X}_\Omega = 1 \end{aligned}$$

Notice that the cross terms vanish in the above computation provided that $p^+up^- = 0$ and $p^+u^*p^- = 0$. Equivalently, we must have that $\varphi^{-1}(y^+) \neq y^-$ and $\varphi(y^+) \neq y^-$ which always holds in the case that both y^+ and y^- come from $V_{0,v_i}(\xi)$.

Therefore, we obtain a unitary $z = (up^+ + p^- u^*)v - (1 - \mathcal{X}_\Omega)$ for every pair of paths from a periodic point to the vertices with the conditions above.

6.3 The Index Pairing Formula

We aim to compute an index pairing of a unitary in $(S \rtimes_{\alpha_s} \mathbb{Z})^\sim$ with the Fredholm module $(S \rtimes_{\alpha_s} \mathbb{Z}, \mathcal{H}, F_e)$ appearing in Proposition 6.1.3. This section gives the formula for the index pairing and appears in [24].

6.3.1 Definition. Let A be a C^*-algebra. The *odd index pairing* between $K_1(A)$ and $K^1(A)$ is the bilinear map $\langle\ ,\ \rangle : K_1(A) \times K^1(A) \to \mathbb{Z}$ defined by the formula $\langle [u], [A, \mathcal{H}, F] \rangle = \text{Index}(P_k U P_k - (1 - P_k))$ where
(i) $(\rho, \mathcal{H}, F)$ is an odd Fredholm module,
(ii) $\mathcal{H}^k$ is the Hilbert space $\mathcal{H}^k = \mathbb{C}^k \otimes \mathcal{H}$,
(iii) u is unitary in a matrix algebra $\mathbb{M}_k(\tilde{A})$ over $\tilde{A}$,
(iv) U is the unitary operator $(1 \otimes \rho)(u)$ on $\mathcal{H}^k$, and
(v) P_k is the operator $1 \otimes \frac{1}{2}(1 + F)$ on $\mathcal{H}^k$

6.3.2 Proposition. *Let A be a C^*-algebra, $(A, \mathcal{H}, F)$ an ungraded Fredholm module over $\tilde{A}$, and u a unitary in $\mathbb{M}_k(\tilde{A})$. Then*

1. *The operator*
$$W = P_k U P_k - (1 - P_k) : \mathcal{H}^k \to \mathcal{H}^k$$
is essentially unitary, and therefore Fredholm,

2. *The index pairing $\langle [u], [A, \mathcal{H}, F] \rangle = \text{Index}(W)$ depends only on the K-theory class of $[u] \in K_1(A)$ and the K-homology class of $[A, \mathcal{H}, F] \in K^1(A)$.*

Using the above proposition in our case gives an index pairing

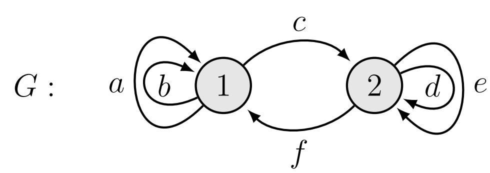
$$K_1(S \rtimes_{\alpha_s} \mathbb{Z}) \times K^1(S \rtimes_{\alpha_s} \mathbb{Z}) \to \mathbb{Z}$$

via the formula

$$\langle [z], [(S \rtimes_{\alpha_s} \mathbb{Z}, \mathcal{H}, F)] \rangle = \text{Index}(\mathscr{P}_e z \mathscr{P}_e - (1 - \mathscr{P}_e))$$

where $z = (up^+ + p^- u^*)v - (1 - \mathcal{X}_\Omega)$ and $\mathscr{P}_e = \frac{1}{2}(F + 1)$ where $(S \rtimes_{\alpha_s} \mathbb{Z}, \mathcal{H}, F)$ is the Fredholm module from Proposition 6.1.3.

6.4 An Example



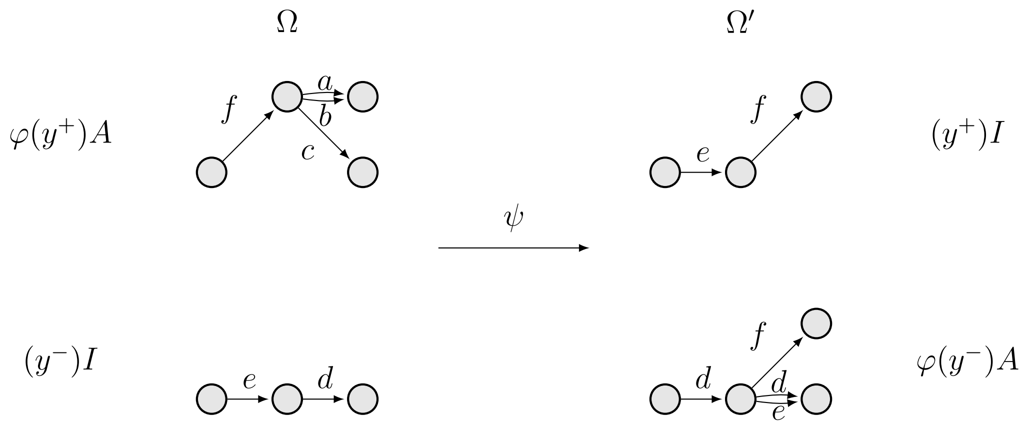
Let (X_G, φ_G) be the shift of finite type arising from the above graph G with adjacency matrix A. From A we have Υ in the kernel of $A - I$ as follows:

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \qquad A - I = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \qquad \Upsilon = \begin{pmatrix} 1 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \end{pmatrix}.$$

We set $Q = \{\cdots eee.eee \cdots\}$ and $P = \{\cdots ddd.ddd \cdots\}$ and the 'special edge' to be the edge c. Now let us define projections using Υ^+ and Υ^-,

$$p^+ = e(e.f, e.f) \qquad\qquad p^- = e(e.d, e.d)$$

where we assume that the fixed point $\cdots eee.eee \cdots$ is extended to the left. Now we obtain sets Ω and Ω':



From the above diagram we read off a bijection $\psi : \Omega \to \Omega'$ giving us a partial isometry

$$v = e(e.f, f.a) + e(d.f, f.b) + e(d.d, f.c) + e(d.e, e.d)$$

Plugging this into the formula for the unitary z we have

$$\begin{aligned} z &= (up^+ + p^- u^*)v - (1 - \mathcal{X}_\Omega) \\ &= (ue(e.f, e.f) + e(e.d, e.d)u^*)v - (1 - \mathcal{X}_\Omega). \end{aligned}$$

Let $W = \mathscr{P}_c z \mathscr{P}_c - (1 - \mathscr{P}_c)$ and we aim to compute the index of W. Observe that for a vector to be in the kernel of W it must be in the kernel of

$$\mathscr{P}_c(ue(e.f, e.f) + e(e.d, e.d)u^*)v\mathscr{P}_c$$

where the above map is from $\mathcal{X}_\Omega \mathscr{P}_c \mathcal{H} \to \mathcal{X}_\Omega \mathscr{P}_c \mathcal{H}$. We shall set up a table to apply these maps with domain $\mathcal{X}_\Omega \mathscr{P}_c \mathcal{H}$:

$(up^+ + p^- u^*)v$	$v^*(p^+ u^* + up^-)$
$f.a \to e.f \to e.f \to ef.$	$f.a \to .fa \to .fa \to f.aa$
$f.b \to d.f \to .df \to .df$	$f.b \to .fb \to .fb \to f.ab$
$f.c \to d.d \to .dd \to .dd$	$f.c \to .fc \to .fc \to f.ac$
$e.d \to d.e \to .de \to .de$	$e.df \to e.df \to ed.f \to f.b$
	$e.dd \to e.dd \to ed.d \to f.c$
	$e.de \to e.de \to ed.e \to e.d$

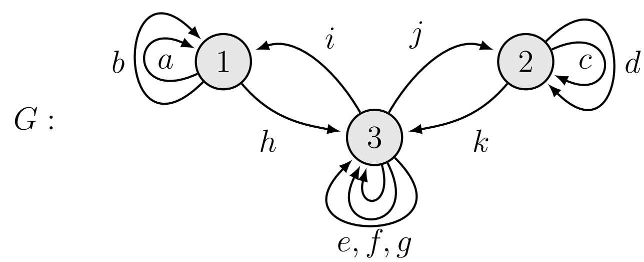
Using the above table we see that the kernel of $\mathscr{P}_c(up^+ + p^- u^*)v\mathscr{P}_c : \mathcal{X}_\Omega\mathscr{P}_c\mathcal{H} \to \mathcal{X}_\Omega\mathscr{P}_c\mathcal{H}$ is trivial while the kernel of $\mathscr{P}_c v^*(p^+u^* + up^-)\mathscr{P}_c : \mathcal{X}_\Omega\mathscr{P}_c\mathcal{H} \to \mathcal{X}_\Omega\mathscr{P}_c\mathcal{H}$ consists of exactly one basis vector; namely,

$$\mathscr{P}_c v^* up^- \mathscr{P}_c \delta_{\cdots eee.ddd\cdots} = \mathscr{P}_c v^* u \delta_{\cdots eee.ddd\cdots} = \mathscr{P}_c v^* \delta_{\cdots eed.ddd\cdots} = \mathscr{P}_c \delta_{\cdots eef.cdd\cdots} = 0.$$

Therefore,

$$\text{Index}(W) = -1.$$

6.5 Another Example



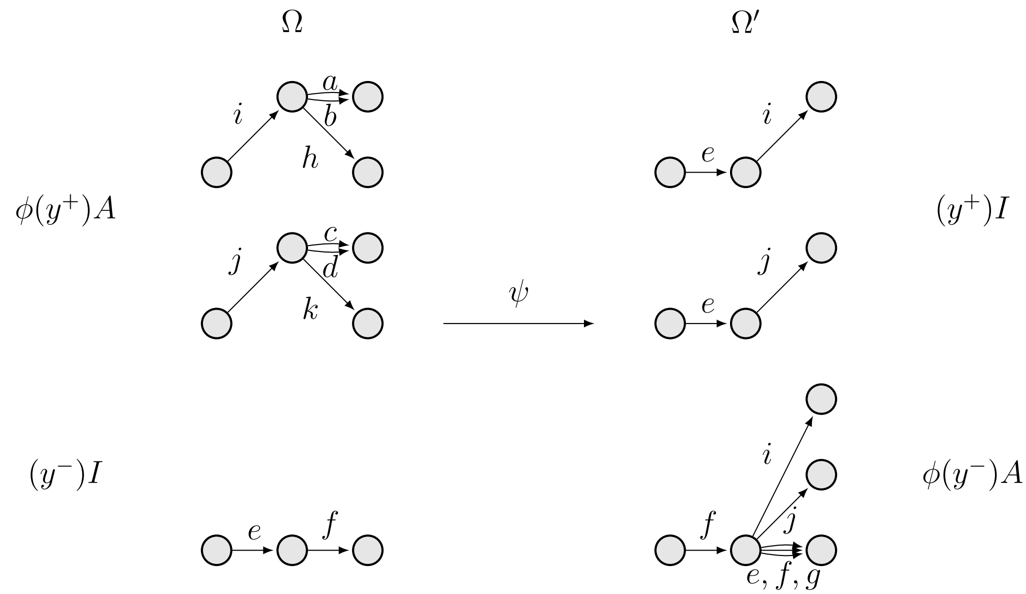
Let (X_G, ϕ) be the shift of finite type arising from the above graph G with adjacency matrix A. From A we have Υ in the kernel of $A - I$ as follows:

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 3 \end{pmatrix} \qquad A - I = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{pmatrix} \qquad \Upsilon = \begin{pmatrix} 1 & 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}.$$

We set $P = \{\cdots eee.eee\cdots\}$ and $Q = \{\cdots fff.fff\cdots\}$ and the 'special edge' to be the edge g. Notice that the graph is symmetric. Now let us define projections using Υ^+ and Υ^-,

$$p^+ = e(e.i, e.i) + e(e.j, e.j) \qquad\qquad p^- = e(e.f, e.f)$$

where we assume that the fixed point is extended to the left. Now we obtain sets Ω and Ω':



From the above diagram we read off a bijection $\psi : \Omega \to \Omega'$:

$$\begin{array}{rcl} i.a & \mapsto & e.i \\ i.b & \mapsto & f.i \\ i.h & \mapsto & f.g \\ j.c & \mapsto & e.j \\ j.d & \mapsto & f.j \\ j.k & \mapsto & f.f \\ e.f & \mapsto & f.e \end{array}$$

which gives us a partial isometry v.

Plugging this into the formula for the unitary z we have

$$\begin{aligned} z &= (up^+ + p^- u^*)v - (1 - \mathcal{X}_\Omega) \\ &= (ue(e.i, e.i) + ue(e.j, e.j) + e(e.f, e.f)u^*)v - (1 - \mathcal{X}_\Omega). \end{aligned}$$

Let $W = \mathscr{P}_g z \mathscr{P}_g - (1 - \mathscr{P}_g)$ and we aim to compute the index of W. Observe that for a vector to be in the kernel of W it must be in the kernel of

$$\mathscr{P}_g(ue(e.i, e.i) + ue(e.j, e.j) + e(e.f, e.f)u^*)v\mathscr{P}_g$$

where the above map is from $\mathcal{X}_\Omega \mathscr{P}_g \mathcal{H} \to \mathcal{X}_\Omega \mathscr{P}_g \mathcal{H}$. We shall set up a table to apply these maps with domain $\mathcal{X}_\Omega \mathscr{P}_g \mathcal{H}$:

$(up^+ + p^- u^*)v$	$v^*(p^+ u^* + up^-)$
$i.a \to e.i \to e.i \to ei.$	$i.a \to .ia \to .ia \to i.aa$
$i.b \to f.i \to .fi \to .fi$	$i.b \to .ib \to .ib \to i.ab$
$i.h \to f.g \to .fg \to .fg$	$i.h \to .ih \to .ih \to i.ah$
$j.c \to e.j \to e.j \to ej.$	$j.c \to .jc \to .jc \to j.cc$
$j.d \to f.j \to .fj \to .fj$	$j.d \to .jd \to .jd \to j.cd$
$j.k \to f.f \to .ff \to .ff$	$j.k \to .jk \to .jk \to j.ck$
$e.f \to f.e \to .fe \to .fe$	$e.fi \to e.fi \to ef.i \to i.b$
	$e.fj \to e.fj \to ef.j \to j.d$
	$e.fe \to e.fe \to ef.e \to e.f$
	$e.ff \to e.ff \to ef.f \to j.k$
	$e.fg \to e.fg \to ef.g \to i.h$

Using the above table we see that the kernel of $\mathscr{P}_g(up^+ + p^- u^*)v\mathscr{P}_g : \mathcal{X}_\Omega \mathscr{P}_g \mathcal{H} \to \mathcal{X}_\Omega \mathscr{P}_g \mathcal{H}$ consists of exactly one basis vector; namely,

$$\mathscr{P}_g p^- u^* v \mathscr{P}_g \delta_{\cdots eei.hff\cdots} = \mathscr{P}_g p^- u^* \delta_{\cdots eef.gff\cdots} = \mathscr{P}_g \delta_{\cdots ee.fgff\cdots} = 0.$$

On the other hand, the kernel of $\mathscr{P}_g v^*(p^+ u^* + up^-)\mathscr{P}_g : \mathcal{X}_\Omega \mathscr{P}_g \mathcal{H} \to \mathcal{X}_\Omega \mathscr{P}_g \mathcal{H}$ is trivial. Therefore,

$$\text{Index}(W) = +1.$$

Chapter 7

Conclusions

Given a Smale space (X, d, φ) with two finite sets of φ-invariant periodic points P and Q, there are four C^*-algebras we have concerned ourselves with. They are the stable algebra $S(X, \varphi, Q)$, the unstable algebra $U(X, \varphi, P)$, the stable Ruelle algebra $S \rtimes_{\alpha_s} \mathbb{Z}$, and the unstable Ruelle algebra $U \rtimes_{\alpha_u} \mathbb{Z}$. In this dissertation we have several results pertaining to the noncommutative geometry of these C^*-algebras.

The first result was to show that $S \rtimes_{\alpha_s} \mathbb{Z}$ and $U \rtimes_{\alpha_u} \mathbb{Z}$ are Poincaré dual, a notion developed by Kasparov and Connes [25, 8]. From the definition of Poincaré duality we obtain isomorphism between the various K-groups of $S \rtimes_{\alpha_s} \mathbb{Z}$ and $U \rtimes_{\alpha_u} \mathbb{Z}$. These isomorphisms are defined by taking the Kasparov product of classes in K-theory and K-homology with the classes δ in $KK^1(\mathbb{C}, S \rtimes_{\alpha_s} \mathbb{Z} \otimes U \rtimes_{\alpha_u} \mathbb{Z})$ and Δ in $KK^1(S \rtimes_{\alpha_s} \mathbb{Z} \otimes U \rtimes_{\alpha_u} \mathbb{Z}, \mathbb{C})$. We defined the class δ using the transverse nature of the stable and unstable equivalence relations and a result of Putnam [34] showing a groupoid equivalence between the product of stable and unstable equivalence and homoclinic equivalence. This gave rise to a Morita equivalence between the C^*-algebras $S(X, \varphi, Q) \otimes U(X, \varphi, P)$ and the homoclinic C^*-algebra. The latter is unitary and K_0 contains the class of the unit, which was used to define δ. The class Δ was more complicated. We defined Δ by producing an extension of $S \rtimes_{\alpha_s} \mathbb{Z} \otimes U \rtimes_{\alpha_u} \mathbb{Z}$ which came from considering various interactions between $S(X, \varphi, Q)$ and $U(X, \varphi, P)$ as well as between $S \rtimes_{\alpha_s} \mathbb{Z}$ and $U \rtimes_{\alpha_u} \mathbb{Z}$. Finally we proved the Duality theorem 4.4.1.

The next set of results concerned spectral triples on the C^*-algebras associated to a Smale space. In particular, we began by defining a type of metric along the stable and unstable equivalence classes of P and Q. These function gave rise to a Dirac operator on the C^*-algebras $S(X, \varphi, Q)$ and $U(X, \varphi, P)$ respectively. Moreover, we showed that the spectral triples were θ-summable and extended to the Ruelle algebras as well. We then observed that the function grew logarithmically with respect to the growth rate of the Smale space which we defined using the entropy. Encoding this information into another function gave a second set of spectral triples on $S(X, \varphi, Q)$ and $U(X, \varphi, P)$ which was finitely summable and the spectral dimension recovered the topological entropy of the Smale space itself.

Finally, we restricted our attention to a shift of finite type and defined a Fredholm module on the stable Ruelle algebra. We then constructed unitaries in $S \rtimes_{\alpha_s} \mathbb{Z}$ using the Pimsner-Voiculescu sequence and paired them with the Fredholm module. We obtained non-zero index in the two examples which we explicitly computed. We note that in one example the index was $+1$ and in the other example the index was -1. These computations give us hope that we will be able to link the Fredholm module we produce with both our Poincaré duality result and our spectral triple results in future work.

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