Self-similar group actions and KMS states

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Self-similar actions

- Suppose X is a finite set of cardinality $|X|$;
  - let X^n denote the set of words of length n in X,
  - let $X^* = \bigcup_{n \in \mathbb{N}} X^n$.

Definition

A faithful action of a group G on X^* is a *self-similar group action* if, for all $g \in G$ and $x \in X$, there exist unique $g|_x \in G$ such that

$$g \cdot (xw) = (g \cdot x)(g|_x \cdot w) \quad \text{for all finite words } w \in X^*.$$

The group element $g|_x$ is called the *restriction* of g to x.

We may replace the letter x by an initial word v: for $g \in G$ and $v \in X^k$ there exists a unique $g|_v \in G$ such that

$$g \cdot (vw) = (g \cdot v)(g|_v \cdot w) \quad \text{for all } w \in X^*.$$

with $\quad g \cdot v = (g \cdot v_1)(g|_{v_1} \cdot v_2) \cdots (g|_{v_1}|_{v_2 \cdots}|_{v_{k-1}} \cdot v_k)$

and $\quad g|_v = (g|_{v_1})|_{v_2} \cdots |_{v_k}$

Restrictions

Lemma

Suppose (G, X) is a self-similar action. Restrictions satisfy

$$g|_{vw} = (g|_v)|_w, \quad gh|_v = g|_{h\cdot v}\ h|_v, \quad g|_v^{-1} = g^{-1}|_{g\cdot v}$$

for all $g, h \in G$ and $v, w \in X^*$.

Proof of $gh|_v = g|_{h\cdot v}\ h|_v$.

For all $w \in X^*$ we have

$$gh \cdot (vw) = (gh \cdot v)(gh|_v \cdot w)$$
$$gh \cdot (vw) = g \cdot (h \cdot v)(h|_v \cdot w) = (gh \cdot v)(g|_{h\cdot v}h|_v \cdot w).$$

Since the action of G on X^* is faithful, we have

$$gh|_v = g|_{h\cdot v}\ h|_v.$$

□

Example: the odometer action

- Let $X = \{0, 1\}$ and $G = \mathbb{Z}$
- Let g denote the generator $1 \in \mathbb{Z}$
- $(\mathbb{Z}, X)$ is a self-similar action described by:

$$g \cdot 0w = 1w \qquad\qquad g \cdot 1w = 0(g \cdot w)$$

  for every finite word $w \in X^*$
- For example, g^3 denotes $3 \in \mathbb{Z}$ and acts on the word 01100 by

$$g^3 \cdot 01100 = g^2 \cdot 11100 = g \cdot 00010 = 10010.$$

The odometer continued

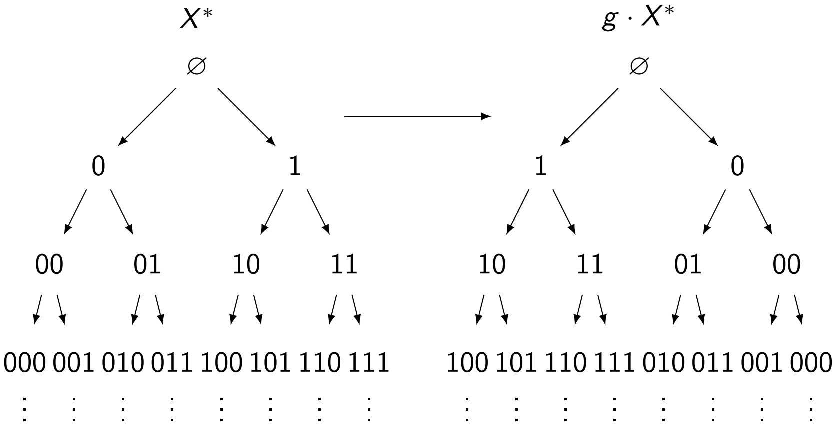


Figure: The odometer action

Contracting SSAs, nucleus and Moore diagrams

- (G, X) is *contracting* if there is a finite $S \subset G$ such that for every $g \in G$ there exists $n \in \mathbb{N}$ with

$$\{g|_v : v \in X^*, |v| \geq n\} \subset S.$$

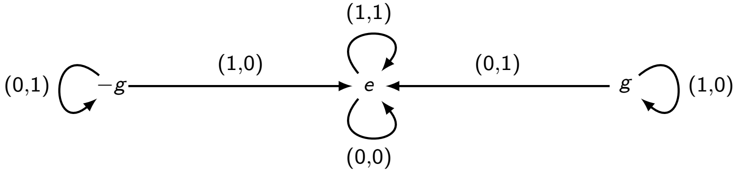
- The *nucleus* of a contracting (G, X) is the smallest such S:

$$\mathcal{N} := \bigcup_{g \in G} \bigcap_{n=0}^{\infty} \{g|_v : v \in X^*, |v| \geq n\}.$$

- For $g \in S$ ($S \subset G$ closed under restriction), the *Moore diagram* with vertex set S has a directed edge $g \xrightarrow{(x,y)} h = g|_x$ for each self similarity relation $g \cdot (xw) = y(h \cdot w)$.

The Moore diagram for the nucleus of the odometer

- The nucleus of the odometer action is $\mathcal{N} = \{e, g, g^{-1}\}$.



The Grigorchuk group (1980)

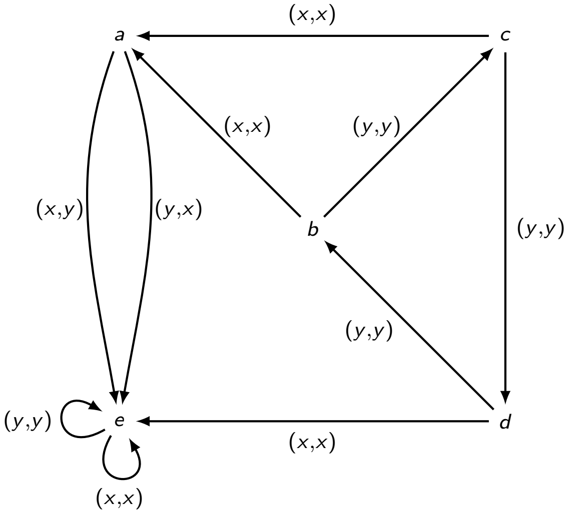
- Let $X = \{x, y\}$
- Consider the rooted homogeneous tree T_X with vertex set X^*.
- Grigorchuk group is generated by four automorphisms a, b, c, d of T_X defined recursively by

$$
\begin{aligned}
a \cdot xw &= yw & \qquad a \cdot yw &= xw \\
b \cdot xw &= x(a \cdot w) & \qquad b \cdot yw &= y(c \cdot w) \\
c \cdot xw &= x(a \cdot w) & \qquad c \cdot yw &= y(d \cdot w) \\
d \cdot xw &= xw & \qquad d \cdot yw &= y(b \cdot w).
\end{aligned}
$$

Proposition

The generators a, b, c, d of G all have order two, and satisfy $cd = b = dc$, $db = c = bd$ and $bc = d = cb$. The self-similar action (G, X) is contracting with nucleus $\mathcal{N} = \{e, a, b, c, d\}$.

The Moore diagram for the nucleus of the Grigorchuk group



Properties of the Grigorchuk group

Theorem (Grigorchuk 1980)

The Grigorchuk group is a finitely generated infinite 2-torsion group.

(This gives a particularly nice example of a Burnside group)

Theorem (Grigorchuk 1984)

The Grigorchuk group has intermediate growth.

(Solved a Milnor problem from 1968)

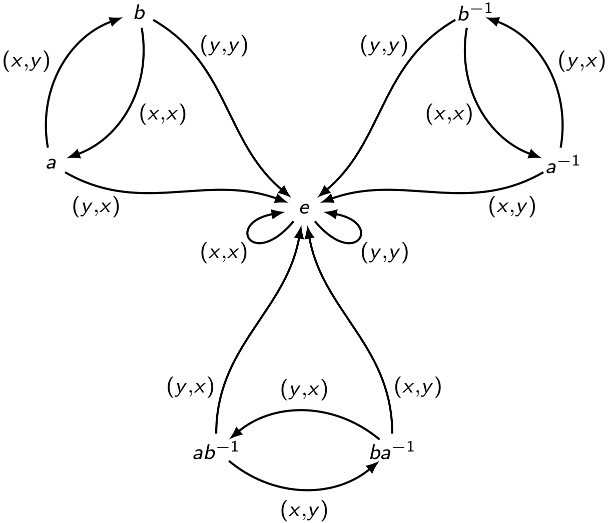
The basilica group [Grigorchuk and Żuk 2003]

- Let $X = \{x, y\}$
- Consider the rooted homogeneous tree T_X with vertex set X^*.
- Two automorphisms a and b of T_X are recursively defined by

$$a \cdot (xw) = y(b \cdot w) \qquad a \cdot (yw) = xw$$
$$b \cdot (xw) = x(a \cdot w) \qquad b \cdot (yw) = yw$$

  for $w \in X^*$.
- The *basilica group* B is the subgroup of $\mathrm{Aut}\, T_X$ generated by $\{a, b\}$. The pair (B, X) is then a self-similar action.
- The nucleus is $\mathcal{N} = \{e, a, b, a^{-1}, b^{-1}, ba^{-1}, ab^{-1}\}$.

The Moore diagram for the nucleus of the basilica group



Properties of the basilica group

Theorem (Grigorchuk and Żuk 2003)

The basilica group

- *is torsion free,*
- *has exponential growth,*
- *has no free non-abelian subgroups,*
- *is not elementary amenable.*

Theorem (Bartholdi and Virág 2005)

The basilica group is amenable.

C^*-algebras

- A C^*-algebra is a Banach $*$-algebra A such that for all a in A,

$$\|aa^*\| = \|a\|^2$$

- Examples: $\mathbb{C}$, $M_n(\mathbb{C})$, $C_0(X)$, $\mathcal{B}(\mathcal{H})$, $\mathcal{K}(\mathcal{H})$, ...

- Any element $U \in A$ such that $U^*U = UU^* = 1$ is called a unitary and any element $S \in A$ such that $S^*S = 1$ is called an isometry.

The C*-algebras $\mathcal{T}(G,X)$ and $\mathcal{O}(G,X)$

Let (G,X) be a self-similar action.

$\mathcal{T}(G,X) :=$ universal C^*-algebra with generators $\{S_x : x \in X\}$ and $\{U_g : g \in G\}$ such that

(T1) $S_y^* S_x = \begin{cases} 1 & \text{if x =y} \\ 0 & \text{if } x \neq y \end{cases}$ $\qquad S \rightsquigarrow \mathcal{T}_{|X|}$

(T2) $U_g U_h = U_{gh}; \quad U_g^* = U_{g^{-1}}; \quad U_e = 1$ $\qquad U \rightsquigarrow C^*(G)$

(T3) $U_g S_x = S_{g\cdot x} U_{g|_x}$ $\qquad$ self-similarity comm. rels. $g\cdot(xw) = (g\cdot x)(g|_x \cdot w)$

$\mathcal{O}(G,X) :=$ the universal Toeplitz algebra $\mathcal{T}(G,X)$ with the extra relation

(O) $\sum_{x\in X} S_x S_x^* = 1$ $\qquad S \rightsquigarrow \mathcal{O}_{|X|}$

Remark: Nekrashevych gave the first definition of $\mathcal{O}(G,X)$.

Spanning set and dynamics

For a word $v = x_1 x_2 \cdots x_n$, we let $S_v := S_{x_1} S_{x_2} \cdots S_{x_n}$.

- $\mathcal{T}(G, X) = \overline{\text{span}}\{S_v U_g S_w^* : v, w \in X^*,\ g \in G\}$.
- If (G, X) is contracting,

$$\mathcal{O}(G, X) = \overline{\text{span}}\{S_v U_g S_w^* : v, w \in X^*,\ g \in \mathcal{N}\}$$

- The dynamics on $\mathcal{T}(G, X)$, and on $\mathcal{O}(G, X)$ are defined by

$$\sigma_t(S_v U_g S_w^*) = e^{t(|v| - |w|)} S_v U_g S_w^*$$

- Interested in (KMS) equilibrium states of $(\mathcal{T}(G, X), \sigma)$ and of $(\mathcal{O}(G, X), \sigma)$.

KMS states

- Given a continuous action $\sigma : \mathbb{R} \to \mathrm{Aut}(A)$, there is a dense *-subalgebra of σ-analytic elements: $t \mapsto \sigma_t(a)$ extends to an entire function $z \mapsto \sigma_z(a)$.

- Definition
  The state φ of A satisfies the KMS condition at inverse temperature $\beta \in [0, \infty)$ if

$$\varphi(ab) = \varphi(b\, \sigma_{i\beta}(a))$$

  whenever a and b are analytic for σ.

- Note: it suffices to verify the above for elements that span a dense subalgebra, e.g, in our case, the spanning set $\{S_v U_g S_w^*\}$

Properties of KMS states:

- Haag-Hugenholtz-Winnink proposed the KMS condition as a definition of equilibrium for quantum systems.
- KMS states are a noncommutative phenomenon, If A has a faithful KMS state and A is commutative, then σ is trivial.
- If $\beta \neq 0$ and φ is a KMS$_\beta$-state, then φ is σ-invariant.
- KMS states have a natural notion of a phase transition (an abrupt change in the physical properties of a system).

Theorem (Laca-Raeburn-Ramagge-W '13)

1. *If* $\beta \in [0, \log|X|)$, *there are no* KMS_β *states for* σ;
2. *if* $\beta \in (\log|X|, \infty]$, *for each normalized trace* τ *on* $C^*(G)$ *define* $\psi_{\beta,\tau}(S_v U_g S_w^*) = 0$ *if* $v \neq w$, *and*

$$\psi_{\beta,\tau}(S_v U_g S_v^*) = (1 - |X|e^{-\beta}) \sum_{k=0}^{\infty} e^{-\beta(k+|v|)} \Bigg(\sum_{\substack{y \in X^k \\ g \cdot y = y}} \tau(\delta_{g|_y}) \Bigg)$$

   the map $\tau \mapsto \psi_{\beta,\tau}$ *is a homeomorphism from the normalised traces on* $C^*(G)$ *to the* KMS_β *states of* $\mathcal{T}(G, X)$.

3. *the* $KMS_{\log|X|}$ *states of* $\mathcal{T}(G, X)$ *arise from KMS states of* $\mathcal{O}(G, X)$; *and there is at least this one:*

$$\psi_{\log|X|}(S_v U_g S_w^*) = \begin{cases} |X|^{-|v|} c_g & \text{if } v = w \\ 0 & \text{otherwise.} \end{cases}$$

   If (G, X) *is contracting, this is the only one.*

The asymptotic proportion of points fixed by $g \in G$

Let τ be the usual trace on $C^*(G)$, i.e. $\tau(\delta_g) = 0$ unless $g = e$; then let $\beta \searrow \log|X|$,

$$\psi_{\beta,\tau}(u_g) = (1 - |X|e^{-\beta}) \sum_{k=0}^{\infty} e^{-\beta k} \Big(\sum_{\substack{y \in X^k \\ g \cdot y = y}} \tau(\delta_{g|_y}) \Big) \longrightarrow \ ??$$

For each $k \in \mathbb{N}$ and $g \in G$ define

$$F_g^k := \{y \in X^n : g \cdot y = y \text{ and } g|_y = e\}.$$

Then $\{|F_g^k||X|^{-k}\} \nearrow c_g \in [0,1)$ and since

$$\sum_{\substack{y \in X^k \\ g \cdot y = y}} \tau(\delta_{g|_y}) = |F_g^k|,$$

the above limit is also c_g.

How do we actually compute c_g ?

Calculating c_g using the Moore diagram

- To calculate values of the KMS states explicitly, we need to evaluate the limit

$$c_g = \lim_{k \to \infty} |F_g^k||X|^{-k}$$

- Each $v \in F_g^k$ corresponds to a path μ_v in the Moore diagram:

$$\mu_v := g \xrightarrow{(v_1, v_1)} g|_{v_1} \xrightarrow{(v_2, v_2)} g|_{v_1 v_2} \xrightarrow{(v_3, v_3)} \cdots \xrightarrow{(v_k, v_k)} g|_v = e$$

- Notice that all the labels have the form (x, x).
- Every path with labels (x, x) arises this way.

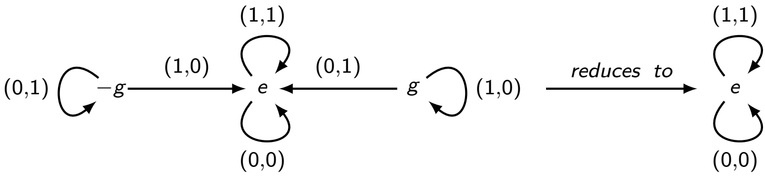
Example: the odometer action

Proposition

The C^*-algebra $\mathcal{O}(\mathbb{Z}, X)$ has a unique $KMS_{\log 2}$ state, which is given on the nucleus $\mathcal{N} = \{e, g, g^{-1}\}$ by

$$\phi(U_n) = \begin{cases} 1 & \text{for } n = e \\ 0 & \text{for } n = g, g^{-1} \end{cases}$$

Sketch.



The Grigorchuk group

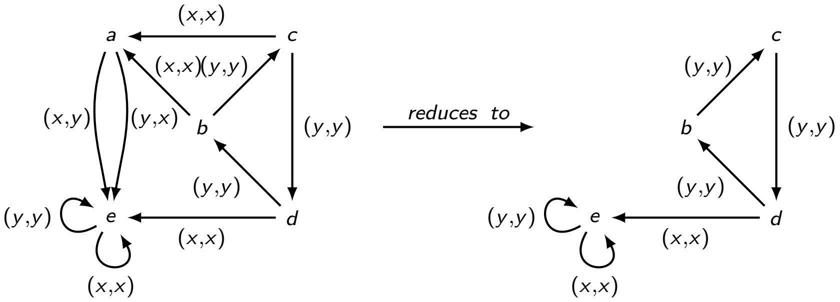
Proposition

Let (G, X) *be the self-similar action of the Grigorchuk group. Then* $(\mathcal{O}(G, X), \sigma)$ *has a unique* $KMS_{\log 2}$ *state* ψ *which is given on the nucleus* $\mathcal{N} = \{e, a, b, c, d\}$ *by*

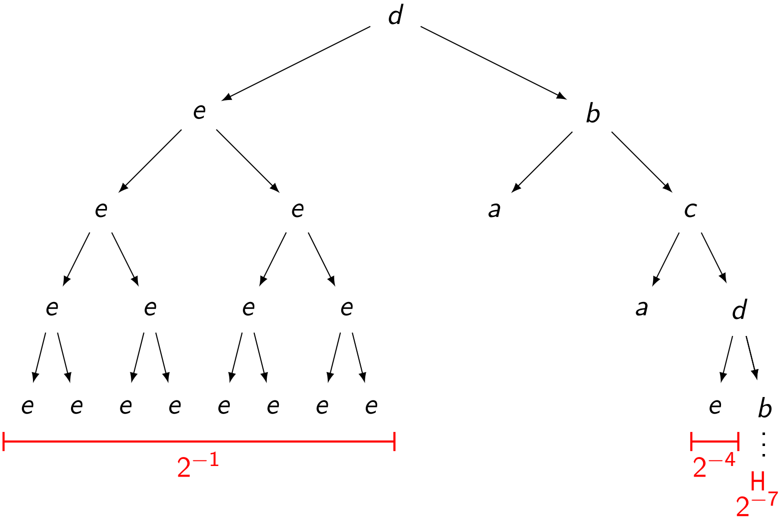
$$\psi(U_g) = \begin{cases} 1 & \text{for } g = e \\ 0 & \text{for } g = a \\ 1/7 & \text{for } g = b \\ 2/7 & \text{for } g = c \\ 4/7 & \text{for } g = d. \end{cases}$$

The Grigorchuk group

Sketch of proof.



Computation of c_d for the Grigorchuk group



$$c_d = \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^{3n} = \frac{1}{2}\left(\frac{1}{1-\frac{1}{8}}\right) = \frac{4}{7}.$$

The basilica group

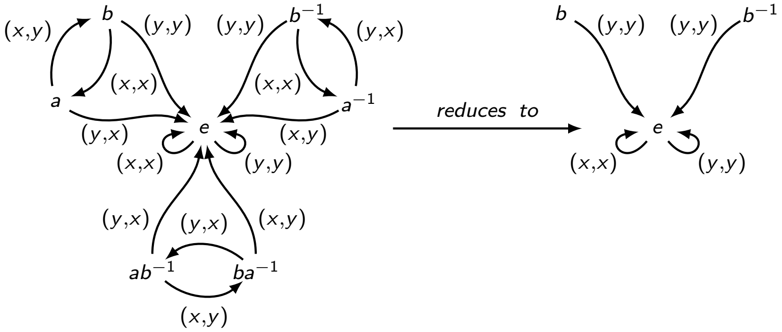
Proposition

The C^*-algebra $\mathcal{O}(B, X)$ has a unique $KMS_{\log 2}$ state, which is given on the nucleus $\mathcal{N} = \{e, a, b, a^{-1}, b^{-1}, ab^{-1}, ba^{-1}\}$ by

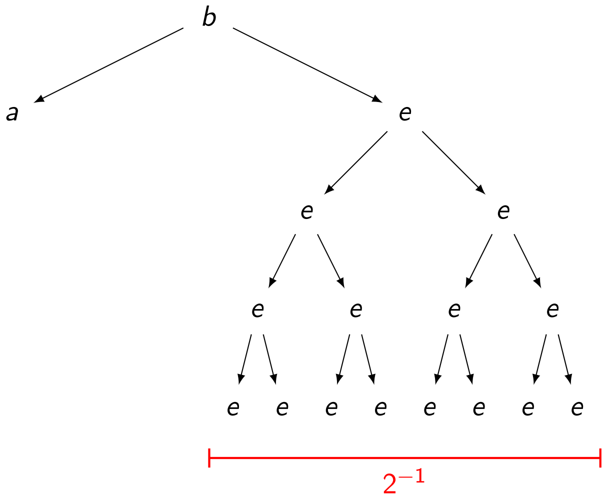
$$\psi(u_g) = \begin{cases} 1 & \text{for } g = e \\ \frac{1}{2} & \text{for } g = b, b^{-1} \\ 0 & \text{for } g = a, a^{-1}, ab^{-1}, ba^{-1}. \end{cases}$$

The basilica group

Sketch of proof.



Computation of c_b for the basilica group



$$c_b = \frac{1}{2}$$

References:

1. M. Laca, I. Raeburn, J. Ramagge, and M. Whittaker *Equilibrium states on the Cuntz-Pimsner algebras of self-similar actions*, ArXiv:1301.4722 (2013).

2. V. Nekrashevych, *C^*-algebras and self-similar groups*, J. reine angew. Math. **630** (2009), 59–123.

3. V. Nekrashevych, *Self-similar groups*, Mathematical Surveys and Monographs **vol. 117**,. Amer. Math. Soc., Providence, RI, 2005.