

# KMS states for self-similar actions

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# Self-similar actions

- Suppose  $X$  is a finite set of cardinality  $|X|$ ;
  - let  $X^n$  denote the set of words of length  $n$  in  $X$ ,
  - let  $X^* = \bigcup_{n \in \mathbb{N}} X^n$ .

## Definition

A faithful action of a group  $G$  on  $X^*$  is *self-similar* if, for all  $g \in G$  and  $x \in X$ , there exist unique  $g|_x \in G$  such that

$$g \cdot (xw) = (g \cdot x)(g|_x \cdot w) \quad \text{for all finite words } w \in X^*.$$

The pair  $(G, X)$  is referred to as a *self-similar action* and the group element  $g|_x$  is called the *restriction* of  $g$  to  $x$ .

# Restrictions

- Restrictions extend to words  $v \in X^*$  in the natural way:

$$g \cdot (vw) = (g \cdot v)(g|_v \cdot w) \quad \text{for all finite words } w \in X^*.$$

## Lemma

*Suppose  $(G, X)$  is a self-similar action. Restrictions satisfy*

$$g|_{pq} = (g|_p)|_q, \quad gh|_p = g|_{h \cdot p} h|_p, \quad g|_p^{-1} = g^{-1}|_{g \cdot p}$$

*for all  $g, h \in G$  and  $p, q \in X^*$ .*

## Example: the odometer action

- Let  $X = \{0, 1\}$  and  $G = \mathbb{Z}$
- Let  $g$  denote the generator  $1 \in \mathbb{Z}$
- $(\mathbb{Z}, X)$  is a self-similar action described by:

$$g \cdot 0w = 1w \qquad g \cdot 1w = 0(g \cdot w)$$

for every finite word  $w \in X^*$

For example,  $g^3$  denotes  $3 \in \mathbb{Z}$  and acts on the word 01100 by

$$g^3 \cdot 01100 = g^2 \cdot 11100 = g \cdot 00010 = 10010.$$

- This defines  $(\mathbb{Z}, X)$  as a self-similar group action called the odometer.

## The odometer continued

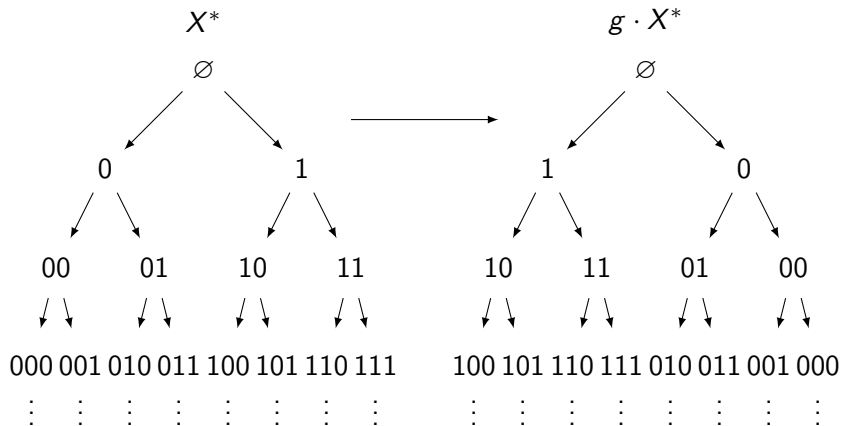


Figure: The action  $g$  on the tree associated with the circle

# Contracting self-similar actions

## Definition

The *nucleus* of a self-similar action  $(G, X)$  is the minimal set  $\mathcal{N} \subset G$  satisfying the property: For every  $g \in G$ , there exists  $N \in \mathbb{N}$  such that  $g|_v \in \mathcal{N}$  for all words  $v \in X^n$  with  $n \geq N$ . A self-similar action  $(G, X)$  is *contracting* if it has a finite nucleus  $\mathcal{N} \subset G$ .

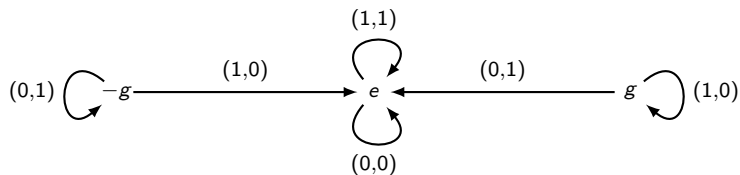
## Definition

Let  $S$  be a subset of  $G$  that is closed under restriction. The *Moore diagram* of  $S$  is the labelled directed graph with vertices in  $S$  and edges labelled:

$$g \xrightarrow{(x,y)} g|_x$$

# The Moore diagram for the nucleus of the odometer

- The nucleus of the odometer action is  $\mathcal{N} = \{e, g, g^{-1}\}$ .



# The basilica group [Grigorchuk and Żuk 2003]

- Let  $X = \{x, y\}$
- Consider the rooted homogeneous tree  $T_X$  with vertex set  $X^*$ .
- Two automorphisms  $a$  and  $b$  of  $T_X$  are recursively defined by

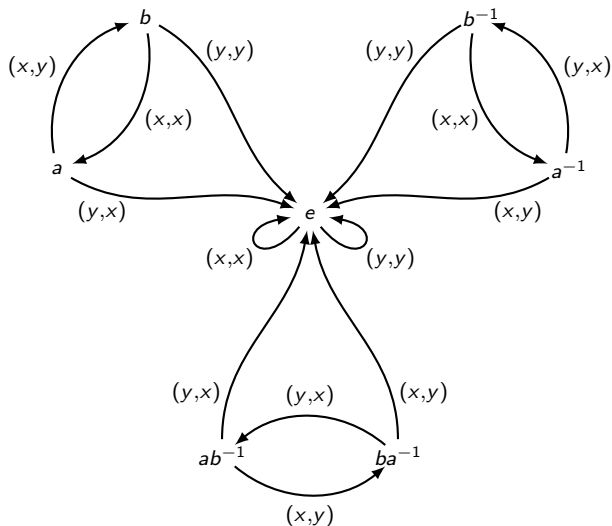
$$\begin{aligned} a \cdot (xw) &= y(b \cdot w) & a \cdot (yw) &= xw \\ b \cdot (xw) &= x(a \cdot w) & b \cdot (yw) &= yw \end{aligned}$$

for  $w \in X^*$ .

- The *basilica group*  $B$  is the subgroup of  $\text{Aut } T_X$  generated by  $\{a, b\}$ . The pair  $(B, X)$  is then a self-similar action.
- The nucleus is  $\mathcal{N} = \{e, a, b, a^{-1}, b^{-1}, ba^{-1}, ab^{-1}\}$ .



# The Moore diagram for the nucleus of the basilica group



# The Grigorchuk action (1980)

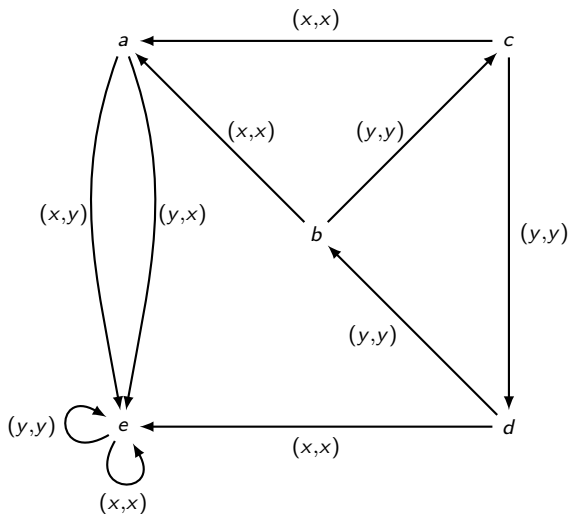
- Let  $X = \{x, y\}$
- Consider the rooted homogeneous tree  $T_X$  with vertex set  $X^*$ .
- Grigorchuk group is generated by four automorphisms  $a, b, c, d$  of  $T_X$  defined recursively by

$$\begin{array}{ll} a \cdot xw = yw & a \cdot yw = xw \\ b \cdot xw = x(a \cdot w) & b \cdot yw = y(c \cdot w) \\ c \cdot xw = x(a \cdot w) & c \cdot yw = y(d \cdot w) \\ d \cdot xw = xw & d \cdot yw = y(b \cdot w). \end{array}$$

The nucleus of the Grigorchuk group is

$$\mathcal{N} = \{e, a, b, c, d\}.$$

# The Moore diagram for the nucleus of the Grigorchuk action



# Universal $C^*$ -algebras for self-similar group actions

## Theorem

Let  $(G, X)$  be a self-similar action. The Cuntz-Pimsner algebra  $\mathcal{O}(G, X)$  is the universal  $C^*$ -algebra generated by unitaries  $\{u_g : g \in G\}$  and a Cuntz family of isometries  $\{s_x : x \in X\}$  satisfying

$$1. \quad u_g s_x = s_{g \cdot x} u_{g|_x}$$

$$2. \quad \sum_{x \in X} s_x s_x^* = 1$$

for all  $g \in G$  and  $x \in X$ . Moreover,

$$\mathcal{O}(G, X) = \overline{\text{span}}\{s_v u_g s_w^* : v, w \in X^*, g \in G\}.$$

## Lemma

When  $(G, X)$  is contracting, the Cuntz-Pimsner algebra  $\mathcal{O}(G, X)$  is finitely generated.

## Proof.

Since  $(G, X)$  is contracting, given  $g \in G$ , there exists  $N$  such that  $g|_v \in \mathcal{N}$  for all  $v \in X^n$  where  $n \geq N$ . The relation (2) from the previous slide implies that

$$1 = \sum_{v \in X^n} s_v s_v^*$$

and we compute

$$u_g = u_g \sum_{v \in X^n} s_v s_v^* = \sum_{v \in X^n} s_{g \cdot v} u_{g|_v} s_v^*$$

and the expression on the right only contains unitaries from the nucleus. □

# KMS states

- Actions of the real line  $\mathbb{R}$  on  $C^*$ -algebras are used to describe time evolution in physics.
  - For a finite quantum system the dynamics is given by a one parameter group of  $*$ -automorphisms of the algebra of observables

$$a \mapsto \sigma_t(a) = e^{itH} a e^{-itH},$$

where  $H$  is the self-adjoint Hamiltonian of the system.

- A state on a  $C^*$ -algebra is a positive linear functional of norm one and we are interested in equilibrium states for the action  $\sigma : \mathbb{R} \rightarrow \text{Aut}(A)$ .

## Definition (Haag-Hughenoltz-Winnink 1967)

Given an action  $\sigma : \mathbb{R} \rightarrow \text{Aut}(A)$  on a  $C^*$ -algebras  $A$ , a state  $\varphi$  satisfies the KMS condition at inverse temperature  $\beta \in [0, \infty)$  if, for all  $a, b \in \mathcal{A}$ ,

$$\varphi(ab) = \varphi(b \sigma_{i\beta}(a)).$$

## Properties of KMS states:

- Haag-Hugenholtz-Winnink proposed the KMS condition as a definition of equilibrium for quantum systems.
- KMS states are a noncommutative phenomenon, If  $A$  has a faithful KMS state and  $A$  is commutative, then  $\sigma$  is trivial.
- If  $\beta \neq 0$  and  $\varphi$  is a  $\text{KMS}_\beta$  state, then  $\varphi$  is  $\sigma$ -invariant.
- KMS states have a natural notion of a phase transition (an abrupt change in the physical properties of a system).

# KMS states for self-similar actions

- An action  $\sigma : \mathbb{R} \rightarrow \text{Aut}(\mathcal{O}(G, X))$  is given by

$$\sigma_t(s_v) = e^{it|v|}s_v \quad \sigma_t(u_g) = u_g.$$

- On the spanning set  $\{s_v u_g s_w^* : v, w \in X^*, g \in G\}$  we have

$$\sigma_t(s_v u_g s_w^*) = e^{it(|v|-|w|)}s_v u_g s_w^*$$



# KMS states on $\mathcal{O}(G, X)$

## Theorem (Laca-Raeburn-Ramagge-W)

Suppose that  $(G, X)$  is a self-similar action.

1. For  $g \in G \setminus \{e\}$ , set

$$F_g^j := \{v \in X^j : g \cdot v = v \text{ and } g|_v = e\}.$$

Then the sequence  $\{|X|^{-j}|F_g^j|\}$  is increasing and converges with limit  $c_g \in [0, 1)$ .

2. There is a  $\text{KMS}_{\log|X|}$  state on  $\mathcal{O}(G, X)$  such that

$$\psi(s_v u_g s_w^*) = \begin{cases} 0 & \text{unless } v = w \\ |X|^{-|w|} c_g & \text{if } v = w. \end{cases}$$

3. If  $(G, X)$  is contracting, then the state in part (2) is the only KMS state of  $\mathcal{O}(G, X)$ .

# Calculating KMS states using the Moore diagram

- To calculate values of the KMS states explicitly, we need to compute the sizes of the sets  $F_g^k$  and evaluate the limit

$$c_g = \lim_{k \rightarrow \infty} |X|^{-k} |F_g^k|$$

- For each  $v \in F_g^k$  we have  $g \cdot v = v$  and  $g|_v = e$ .
- Each  $v \in F_g^k$  corresponds to a path  $\mu_v$  in the Moore diagram:

$$\mu_v := g \xrightarrow{(v_1, v_1)} g|_{v_1} \xrightarrow{(v_2, v_2)} g|_{v_1 v_2} \xrightarrow{(v_3, v_3)} \cdots \xrightarrow{(v_k, v_k)} g|_v = e$$

- Notice that all the labels have the form  $(x, x)$ .
- Every path with labels  $(x, x)$  arises this way.

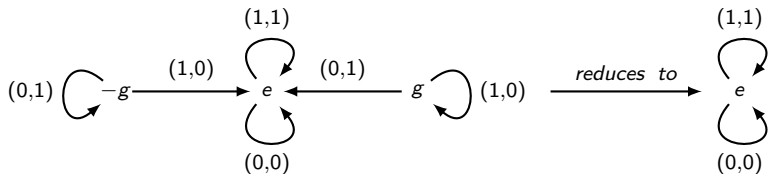
# Example: the odometer action

## Proposition

The  $C^*$ -algebra  $\mathcal{O}(\mathbb{Z}, X)$  has a unique  $KMS_{\log 2}$  state, which is given on the nucleus  $\mathcal{N} = \{e, g, g^{-1}\}$  by

$$\phi(u_n) = \begin{cases} 1 & \text{for } n = e \\ 0 & \text{for } n = g, g^{-1} \end{cases}$$

*Sketch.*



## Example: the basilica action

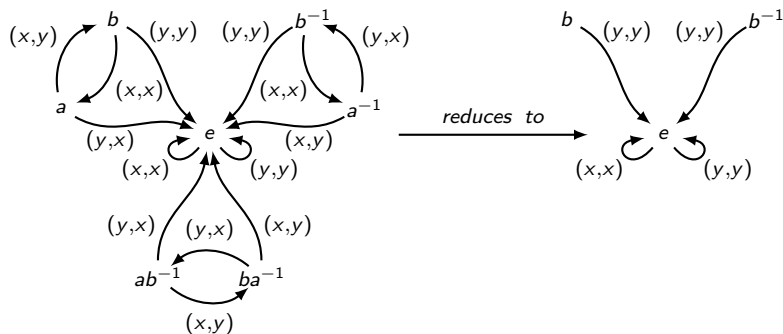
### Proposition

The  $C^*$ -algebra  $\mathcal{O}(B, X)$  has a unique  $KMS_{\log 2}$  state, which is given on the nucleus  $\mathcal{N} = \{e, a, b, a^{-1}, b^{-1}, ab^{-1}, ba^{-1}\}$  by

$$\phi(u_g) = \begin{cases} 1 & \text{for } g = e \\ \frac{1}{2} & \text{for } g = b, b^{-1} \\ 0 & \text{for } g = a, a^{-1}, ab^{-1}, ba^{-1}. \end{cases}$$

## Example: the basilica action

Sketch.



## Example: the Grigorchuk action

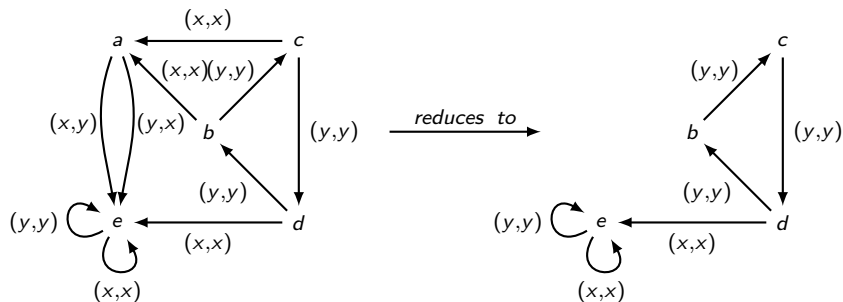
### Proposition

Let  $(G, X)$  be the self-similar action of the Grigorchuk group. Then  $(\mathcal{O}(G, X), \sigma)$  has a unique  $KMS_{\log 2}$  state  $\phi$  which is given on the nucleus  $\mathcal{N} = \{e, a, b, c, d\}$  by

$$\phi(u_g) = \begin{cases} 1 & \text{for } g = e \\ 0 & \text{for } g = a \\ 1/7 & \text{for } g = b \\ 2/7 & \text{for } g = c \\ 4/7 & \text{for } g = d. \end{cases}$$

# Example: the Grigorchuk action

*Sketch.*



## References:

1. M. Laca, I. Raeburn, J. Ramagge, and M. Whittaker *Equilibrium states on the Cuntz-Pimsner algebras of self-similar actions*, ArXiv:1301.4722 (2013).
2. V. Nekrashevych,  *$C^*$ -algebras and self-similar groups*, J. reine angew. Math. **630** (2009), 59–123.
3. V. Nekrashevych, *Self-similar groups*, Mathematical Surveys and Monographs **vol. 117**,. Amer. Math. Soc., Providence, RI, 2005.