Topological spaces associated to higher-rank graphs

AustMS 2013
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1 October 2013

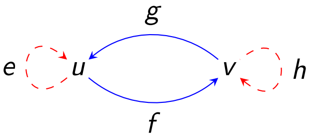
Plan

1. Introduction to k-graphs via coloured graphs;
2. Topological realisations of k-graphs;
3. Surfaces and the connected-sum operation;
4. Higher dimensional simplices and spheres.

Coloured graphs

A *coloured directed graph* $E = (E^0, E^1, r, s, c)$ consists of a vertex set E^0 an edge set E^1, range and source maps $r, s : E^1 \to E^0$ and a colour map $c : E^1 \to \{c_1, \ldots, c_k\}$ where we think of $\{c_1, \ldots, c_k\}$ as k different colours.

Example



$E^0 = \{v, u\}$
$v = s(g) = s(h) = r(h) = r(f)$
$c(f) = c(g) = c_1\,(= blue)$

$E^1 = \{e, f, g, h\}$
$u = s(f) = s(e) = r(e) = r(g)$
$c(e) = c(h) = c_2\,(= red)$.

k-graphs

- A *path* of edges is a sequence $\mu = \mu_1\mu_2\mu_3 \cdots \mu_n$ such that $s(\mu_i) = r(\mu_{i+1})$ for all $i = 1, \cdots, n$.
- We define $r(\mu) = r(\mu_1)$ and $s(\mu) = s(\mu_n)$ as depicted:

$$r(\mu) \xleftarrow{\ \mu_1\ } \cdots\cdots\cdots \xleftarrow{\ \mu_n\ } s(\mu)$$

- A vertex that only emits edges is called a *source* and a vertex that only receives edges is called a *sink*.

Definition (Kumjian and Pask, New York J. Math. 2000)

A *k-graph* Λ is a countable small category with a degree functor $d : \Lambda \to \mathbb{N}^k$ satisfying the *factorisation property*: if $\lambda \in \mathrm{Mor}(\Lambda)$ has degree $d(\lambda) = m + n$, then there exists unique $\mu, \nu \in \mathrm{Mor}(\Lambda)$ with $d(\mu) = m$ and $d(\nu) = n$ with $\lambda = \mu\nu$.

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Definition

Suppose Λ is a k-graph and let $\{e_1, \cdots, e_k\}$ denote the generators of $\mathbb{N}^k$. The *skeleton* of Λ is the k-coloured directed graph E_Λ with vertex set $E^0 = \text{Obj}(\Lambda)$, edge set $E^1 = \bigcup_{i=1}^k d^{-1}(e_i)$ with colouring map $c : E_\Lambda^1 \to \{c_1, \cdots, c_k\}$ given by $c(f) = c_i \iff d(f) = e_i$. Note that edges inherit range and source maps (since they are morphisms). In addition, the factorisation property determines a set of *factorisation rules:* $fg = g'f'$ whenever $f, f' \in c^{-1}(c_i)$, $g, g' \in c^{-1}(c_j)$, and $fg = g'f' \in \Lambda$.

- Every k-graph has a skeleton. Wouldn't it be nice to go the other way?

Skeletons have k-graphs in the closet

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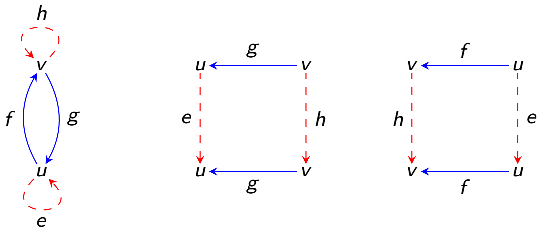
Theorem (Hazlewood-Raeburn-Sims-Webster)

Suppose $E = (E^0, E^1, r, s, c)$ is a coloured graph and for each distinct $i, j \leq k$ there is a range and source preserving bijection θ_{ij} between $c_i c_j$-coloured paths and $c_j c_i$-coloured paths satisfying an associativity condition. Let $\sim$ be the smallest equivalence relation on the path space E^* generated by $fg \sim g'f'$ if and only if $\theta_{ij}(fg) = g'f'$. Then the quotient space $E^*/\sim$ is a k-graph.

Example

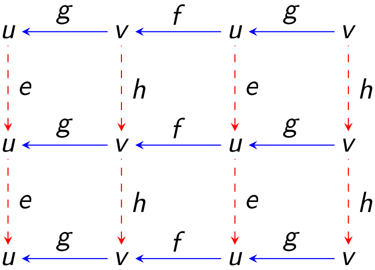
Let E be the coloured graph depicted below. We define the bijections θ_{ij} by "commuting squares."



The squares above determine a bijection from blue-red edges to red-blue edges via $gh \mapsto eg$ and $fe \mapsto hf$

Example (continued)

This is the way we view a degree $(3, 2)$ path in the k-graph.

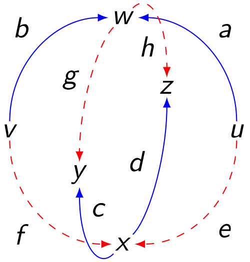


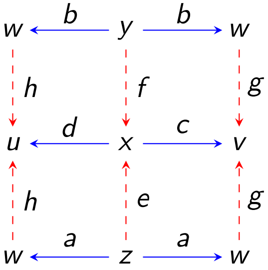
The equivalence relation implies that any path from the top right vertex to the bottom left vertex is equivalent.

Topological realisations of k-graphs

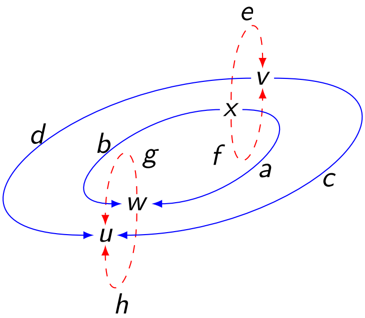
- In the 2012 preprint *Topological realisations and fundamental groups of higher-rank graphs* by Kaliszewski, Kumjian, Quigg, and Sims, the topological realisation of a k-graph was developed.
- Given a k-graph Λ, denote the topological realisation by X_Λ.
- They showed that the topological realisation of a 2-graph is homeomorphic to pasting a unit square into each commuting square and then identifying all instances of vertices and edges that appear more than once.
- In particular, they were able to construct 2-graphs for the four fundamental surfaces: the sphere, torus, Klein bottle, and projective plane.
- The skeletons and planar diagrams of these surfaces appear on the following four slides.

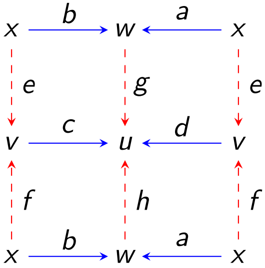
The sphere



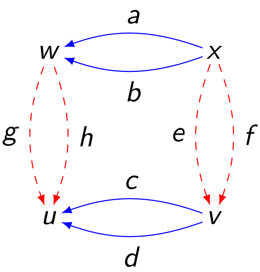


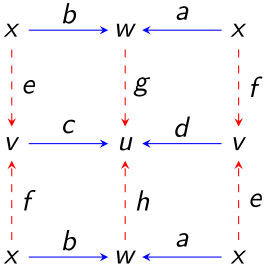
The torus



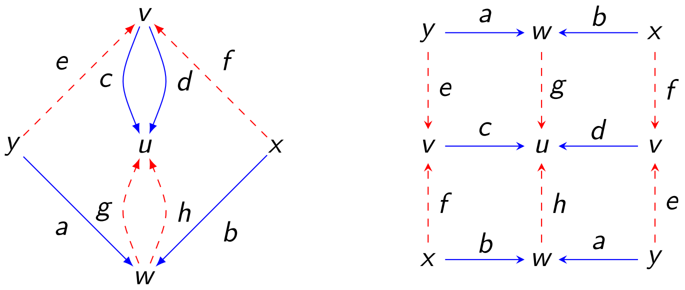


The Klein bottle





The projective plane



The connected-sum

- The topological connected-sum of two surfaces is given by removing an open disk from each surface and glueing the two surfaces together along the boundaries of the disks.
- Let S_1 and S_2 be surfaces, the connected-sum is denoted $S_1 \# S_2$.

Theorem

Every surface (compact 2-manifold) is a connected sum of a finite number of copies of the four basic surfaces.

- In order to realise every surface as a topological realisation of a 2-graph we need to develop a notion of connected sum for 2-graphs.

The 2-graph connected-sum

Lemma

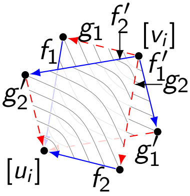
Suppose Λ_1 and Λ_2 are 2-graphs and $u_i, v_i \in \Lambda_i^0$ have the property that both u_i's are sinks and both v_i's are sources for $i = 1, 2$. Let $\sim$ be the smallest equivalence relation on $\Lambda_1 \sqcup \Lambda_2$ such that $u_1 \sim u_2$ and $v_1 \sim v_2$. Then the quotient $(\Lambda_1 \sqcup \Lambda_2)/\sim$ is a 2-graph.

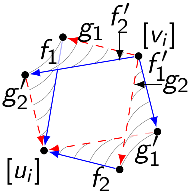
- In the 2-graph $(\Lambda_1 \sqcup \Lambda_2)/\sim$ we assume that we can find commuting squares in Λ_1 and Λ_2 with source $[v_i]$ and range $[u_i]$.
- We will remove the interior of these squares and replace them with commuting squares connecting Λ_1 and Λ_2.
- A picture should help...

The 2-graph connected-sum

$f_1 g_1 = g_1' f_1'$

$f_2 g_2 = g_2' f_2'$





$f_1 g_1 = g_2' f_2'$

$f_2 g_2 = g_1' f_1'$

The above operation deletes the interior of the square $f_1 g_1 = g_1' f_1'$ in Λ_1 and the square $f_2 g_2 = g_2' f_2'$ in Λ_2, and inserts a copy of a unit square bounded by f_1, g_1, f_2' and g_2' and another bounded by f_1', g_1', f_2 and g_2.

The 2-graph connected-sum

Lemma

The operation on the previous page defines a 2-graph, denoted $\Lambda_1 \# \Lambda_2$ *such that*

$$X_{\Lambda_1 \# \Lambda_2} \cong X_{\Lambda_1} \# X_{\Lambda_2}.$$

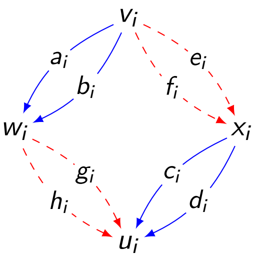
The above process can be iterated, so we obtain the following theorem:

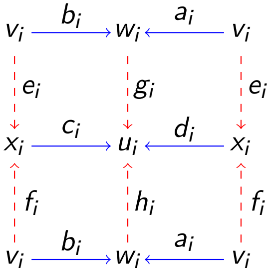
Theorem (Kumjian, Pask, Sims, and W)

For each compact 2-dimensional manifold M*, there is a 2-graph* Λ *such that* $X_\Lambda \cong M$*.*

Example: the two holed torus

Consider two copies of the 2-graph torus Λ_1 and Λ_2. The skeleton and commuting squares are given below (where edge $e_i \in \Lambda_i$)

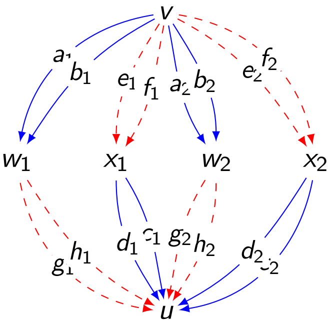


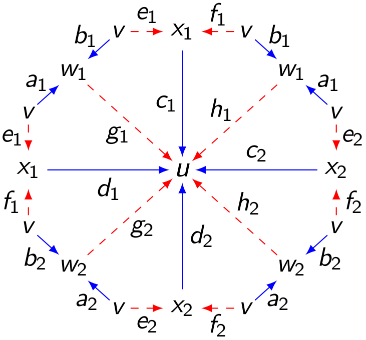


To apply our construction we set $u = [u_i]$ and $v = [v_i]$ with the distinguished squares $d_1 f_1 = h_1 a_1$ of Λ_1 and $c_2 e_2 = g_2 b_2$ of Λ_2.

Example continued

Then the skeleton of the connected sum $\Lambda_1 \# \Lambda_2$ has the form of the diagram on the left with the replaced factorisation rules $d_1 f_1 = g_2 b_2$ and $h_1 a_1 = c_2 e_2$.





Organising this skeleton into the diagram on the right, we recognise the standard planar diagram for a two-holed 2-torus.

What about higher dimensions?

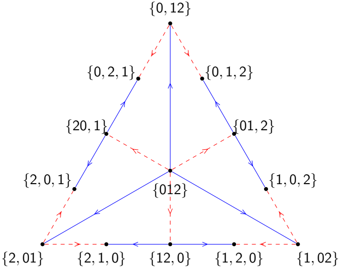
- We began with the aim of realising every simplicial manifold as the topological realisation of a k-graph.
- We were able to construct simplices in every dimension but the rigid combinatorics of k-graphs precluded us from gluing them together arbitrarily.
- However, we were able to realise every k-sphere as the topological realisation of a k-graph.

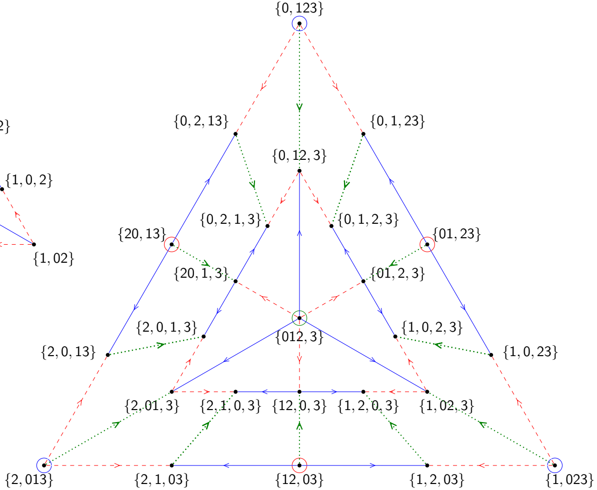
Theorem (Kumjian, Pask, Sims, and W)

For each $k \geq 0$ there is a finite k-graph Λ whose topological realisation is homeomorphic to a k-sphere.

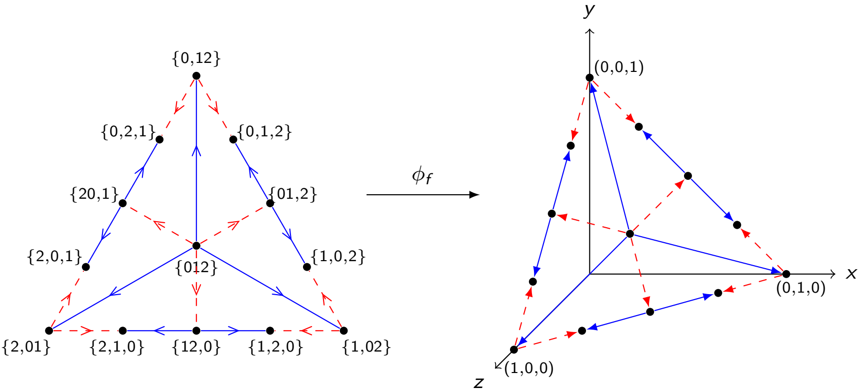
Simplices

1, 1, 3, 13, 75, 541, 4683, 47293, ...





Realising a k-simplex in $\mathbb{R}^{k+1}$



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